

Contraction Obstructions for Connected Graph Searching[☆]

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Abstract

We consider the connected variant of the classic mixed search game where, in each search step, cleaned edges form a connected subgraph. We consider graph classes with bounded connected (and monotone) mixed search number and we deal with the question whether the obstruction set, with respect of the contraction partial ordering, for those classes is finite. In general, there is no guarantee that those sets are finite, as graphs are not well quasi ordered under the contraction partial ordering relation. In this paper we provide the obstruction set for $k = 2$, where k is the number of searchers we are allowed to use. This set is finite, it consists of 177 graphs and completely characterises the graphs with connected (and monotone) mixed search number at most 2. Our proof reveals that the “sense of direction” of an optimal search searching is important for connected search which is in contrast to the unconnected original case. We also give a double exponential lower bound on the size of the obstruction set for the classes where this set is finite.

Keywords: Graph Searching, Graph Contractions, Obstruction set

1. Introduction

A *mixed searching game* is defined in terms of a graph representing a system of tunnels where an agile and omniscient fugitive with unbounded speed is hidden (alternatively, we can formulate the same problem considering that the

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tunnels are contaminated by some poisonous gas). The fugitive is occupying the edges of the graph and the searchers can be placed on its vertices. In the beginning of the game, the fugitive chooses some edge and there are no searchers at all on the graph. The objective of the searchers is to deploy a search strategy on the graph that can guarantee the capture of the fugitive. The fugitive is *captured* if at some point he resides on an edge e and one of the following capturing cases occurs.

A: *both endpoints of e are occupied by a searcher,*

B: *a searcher slides along e , i.e., a searcher is moved from one endpoint of the edge to the other endpoint.*

A *search strategy* on a graph G is a finite sequence \mathcal{S} containing moves of the following types.

$p(v)$: placing a new searcher on a vertex v ,

$r(v)$: deleting a searcher from a vertex v ,

$s(v, u)$: sliding a searcher on v along the edge $\{v, u\}$ and placing it on u .

We stress that the fugitive is *agile* and *omniscient*, i.e. he moves at any time in the most favourable, for him, position and is *invisible*, i.e. the searchers strategy is given “in advance” and does not depend on the moves of the fugitive during it.

Given a search \mathcal{S} , we denote by $E(\mathcal{S}, i)$ the set of edges that are clean after applying the first i steps of \mathcal{S} , where by “clean” we mean that the search strategy can guarantee that none of its edges will be occupied by the fugitive after the i -th step. More formally, we set $E(\mathcal{S}, 0) = \emptyset$ and in step $i > 0$ we define $E(\mathcal{S}, i)$ as the set defined as follows: first consider the set Q_i containing all the edges in $E(\mathcal{S}, i - 1)$ plus the edges of $E^{(i)}$ the set of edges that are cleaned after the i -th move because of the application of cases **A** or **B**. Notice that $E^{(i)}$ may be empty. In particular, it may be non-empty in case the i -move is a placement move, will always be empty in case the i -th move is a removal move and will surely be non-empty in case the i -th move is a sliding move. In the third case, the edge along which the sliding occurs is called *the sliding edge* of $E^{(i)}$. Then, the set $E(\mathcal{S}, i)$ is defined as the set of all edges in Q_i minus those for which there is a path starting from them and finishing in an edge not in Q_i . This expresses the fact that the agile and omniscient fugitive could use any of these paths in order to occupy again some of the edges in Q_i . In case $E(\mathcal{S}, i) \subset Q_i$, we say that the i -th move is a *recontamination move*. Notice that in such a case we have that $E(\mathcal{S}, i - 1) \not\subseteq E(\mathcal{S}, i)$.

The object of a mixed search is to clear all edges using a search. We call search \mathcal{S} *complete* if at some step all edges of G are clean, i.e. $E(\mathcal{S}, i) = E(G)$ for some i .

Connected monotone mixed search number. The mixed search number of a search is the maximum number of searchers on the graph during any move. A search without recontamination moves is called *monotone*. Mixed search number has been introduced in [1]. The mixed search number, $\mathbf{ms}(G)$, of a graph G is the minimum mixed search number over all the possible complete searches on it (if G is an edgeless graph, then this number is 0). A search is *connected* if $E(\mathcal{S}, i)$ induces a connected subgraph of G for every step i . Given a graph G , we will denote the minimum mixed search number over all the possible complete connected searches on it by $\mathbf{cms}(G)$ and we will call this number connected mixed search number of G . The monotone (resp. connected monotone) mixed search number, $\mathbf{mms}(G)$ (resp. $\mathbf{cmms}(G)$), of G is the minimum mixed search number over all the possible complete monotone (connected monotone) searches of it (connected variants are defined only under the assumption that G is a connected graph). The concept of connectivity in graph searching was introduced for the first time in [2] and was motivated by application of graph searching where the “clean” territories should be maintained connected so to guarantee the safe communication between the searchers during the search.

Obstructions. Given a graph invariant \mathbf{p} , a partial ordering relation on graphs \trianglelefteq , and an integer k we denote by $\mathbf{obs}_{\trianglelefteq}(\mathcal{G}[\mathbf{p}, k])$ the set of all \trianglelefteq -minimal graphs G where $\mathbf{p}(G) > k$ and we call it *the k -th \trianglelefteq -obstruction set for \mathbf{p}* . We also say that \mathbf{p} is *closed under \trianglelefteq* if for every two graphs H and G , $H \trianglelefteq G$ implies that $\mathbf{p}(H) \leq \mathbf{p}(G)$. Clearly, if \mathbf{p} is closed under \trianglelefteq , then the k -th \trianglelefteq -obstruction set for \mathbf{p} provides a complete characterisation for the class $\mathcal{G}_k = \{G \mid \mathbf{p}(G) \leq k\}$: a graph belongs in \mathcal{G}_k iff none of the graphs in the k -th \trianglelefteq -obstruction set for \mathbf{p} is contained in G with respect to the relation \trianglelefteq .

Our results. In this paper we are interested in obstruction characterisations for the graphs with bounded connected (monotone) mixed search number. While it is known that \mathbf{ms} is closed under taking of minors, this is not the case for \mathbf{cms} and \mathbf{cmms} where the connectivity requirement applies. From Robertson and Seymour Theorem [3], the k -th \leq -obstruction set for \mathbf{ms} is always finite, where \leq is the minor partial ordering relation (defined formally in Subsection 2.4). Moreover this set has been found for $k = 1$ (2 graphs) and $k = 2$ (36 graphs) in [4]. However, no such result exists for the obstruction characterisations of the connected monotone mixed search number. As we prove in this paper, \mathbf{cms} and \mathbf{cmms} are closed under contractions. Unfortunately, graphs are not well quasi ordered with respect to the contraction relation, therefore there is no guarantee that the k -th contraction obstruction set for \mathbf{cms} or \mathbf{cmms} is finite for all k . The finiteness of this set is straightforward if $k = 1$ as $\mathbf{obs}_{\preceq}(\mathcal{G}[\mathbf{cmms}, 1]) = \{K_3, K_{1,3}\}$. In this paper we completely resolve the case where $k = 2$. We prove that $\mathbf{obs}_{\preceq}(\mathcal{G}[\mathbf{cmms}, 2]) = \mathbf{obs}_{\preceq}(\mathcal{G}[\mathbf{cms}, 2])$ and we prove that this set is finite by providing all 177 graphs that it contains. The proof of our results is based on a series of lemmata that confine the structure of the graphs with connected monotone mixed search number at most 2. We should stress that, in contrary to the case of \mathbf{ms} the direction of searching is crucial for \mathbf{cmms} . This

makes the detection of the corresponding obstruction sets more elaborated as special obstructions are required in order to force a certain sense of direction in the search strategy. For this reason, our proof makes use of a more general variant of the mixed search strategy that forces the searchers to start and finish to specific sets of vertices. Obstructions for this more general type of searching are combined in order to form the required obstructions for **cmms**. We also give a double exponential lower bound on the size of the contraction obstruction set for the classes with bounded connected monotone search number. This lower bound is only meaningful for the classes where this obstruction set is finite.

2. Preliminary Definitions and Results

Let A be a set and let $\mathcal{A} = \langle a_1, \dots, a_r \rangle$ be an ordering of A . We denote by $\text{prefsec}(\mathcal{A})$ the ordering $\langle A_0, \dots, A_r \rangle$ of subsets of A , where $A_0 = \emptyset$ and for $i = 1, \dots, r$, $A_i = \{a_1, \dots, a_i\}$. Let \mathcal{A}_1 and \mathcal{A}_2 be two disjoint orderings of A , we denote by $\mathcal{A}_1 \oplus \mathcal{A}_2$ the concatenation of these two orderings.

All graphs under consideration will be finite, without loops or multiple edges. Let G be a graph and $e = \{u, v\} \in E(G)$ an edge. The *edge-contraction* of $\{u, v\}$ (or just the *contraction* of $\{u, v\}$) is the operation that deletes this edge, adds a new vertex x_{uv} and connects this vertex to all the neighbours of u and v (if some multiple edges are created we delete them). We denote by G/e the graph obtained from G by contracting edge e .

If $S \subseteq V(G)$ we call graph $G[S] = (S, \{\{u, v\} \in E(G) \mid u, v \in S\})$ the *subgraph of G induced by S* . Also, given a set $F \subseteq E(G)$ we call graph $G[F] = (\bigcup_{e \in F} e, F)$ the *subgraph of G induced by F* and we denote by $V(F)$ the set of vertices in $G[F]$.

We define the *union* of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ to be the graph $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. When V_1 and V_2 are disjoint, we refer to this union as the *disjoint union* of G_1 and G_2 (we denote it as $G_1 + G_2$).

A vertex of a graph is called *pendant* if it has degree at most 1. An edge e of a graph G is *pendant* if one of its endpoints is pendant. If both endpoints of an edge of G are pendant, then we say that e is an isolated edge.

We adapt the standard notations for the neighbourhood and the degree of a vertex $u \in V(G)$, i.e. the set of all vertices connected with u by an edge and the cardinality of this set, which is $N_G(u)$ and $\deg_G(u)$ respectively.

2.1. Rooted graph triples.

A *rooted graph triple*, or, for simplicity, a *rooted graph*, is an ordered triple $(G, S^{\text{in}}, S^{\text{out}})$ where G is a connected graph and S^{in} and S^{out} are subsets of $V(G)$ (S^{in} and S^{out} are not necessarily disjoint sets). If $\mathbf{G} = (G, S^{\text{in}}, S^{\text{out}})$ then we also say that \mathbf{G} is the graph G *in-rooted* on S^{in} and *out-rooted* at S^{out} . Given a rooted graph $\mathbf{G} = (G, S^{\text{in}}, S^{\text{out}})$, we define $\text{rev}(\mathbf{G}) = (G, S^{\text{out}}, S^{\text{in}})$.

Given a rooted graph $(G, S^{\text{in}}, S^{\text{out}})$, where

$$S^{\text{in}} = \{v_1^{\text{in}}, \dots, v_{|S^{\text{in}}|}^{\text{in}}\} \quad \text{and} \quad S^{\text{out}} = \{v_1^{\text{out}}, \dots, v_{|S^{\text{out}}|}^{\text{out}}\},$$

we define its *enhancement* as the graph $\mathbf{enh}(G, S^{\text{in}}, S^{\text{out}})$ obtained from G after adding two vertices u^{in} and u^{out} and the edges in the sets

$$E^{\text{in}} = \{\{v_1^{\text{in}}, u^{\text{in}}\}, \dots, \{v_{|S^{\text{in}}|}^{\text{in}}, u^{\text{in}}\}\}$$

and

$$E^{\text{out}} = \{\{v_1^{\text{out}}, u^{\text{out}}\}, \dots, \{v_{|S^{\text{out}}|}^{\text{out}}, u^{\text{out}}\}\}.$$

From now on, we will refer to the vertices $u^{\text{in}}, u^{\text{out}}$ as the *vertex extensions* of $\mathbf{enh}(G, S^{\text{in}}, S^{\text{out}})$ and the edge sets E^{in} and E^{out} as the *edge extensions* of $\mathbf{enh}(G, S^{\text{in}}, S^{\text{out}})$.

2.2. An extension of the connected search game.

In the above setting we assumed that searchers cannot make their first move in the graph before the fugitive makes his first move. Let G be a graph and let $S^{\text{in}}, S^{\text{out}} \subseteq V(G)$. A $(S^{\text{in}}, S^{\text{out}})$ -complete strategy for G is a search strategy \mathcal{S} on $\mathbf{enh}(G, S^{\text{in}}, S^{\text{out}})$ such that

- (i) $E(\mathcal{S}, i) = E^{\text{in}}$, for some i ,
- (ii) $E(\mathcal{S}, i) \cap E^{\text{out}} = \emptyset$, for every i and
- (iii) $E(\mathcal{S}, i) = E(G) \setminus E^{\text{out}}$, for some i ,

where $E^{\text{in}}, E^{\text{out}}$ are the *edge extensions* of $\mathbf{enh}(G, S^{\text{in}}, S^{\text{out}})$.

Based on the above definitions, we define $\mathbf{ms}(G, S^{\text{in}}, S^{\text{out}})$ as the minimum mixed search number over all possible $(S^{\text{in}}, S^{\text{out}})$ -complete search strategies for it. Similarly, we define $\mathbf{mms}(G, S^{\text{in}}, S^{\text{out}})$ and $\mathbf{cmms}(G, S^{\text{in}}, S^{\text{out}})$ where, in the case of connected searching, we additionally demand that S^{in} induces a connected subgraph of G . Notice that $\mathbf{ms}(G) = \mathbf{ms}(G, \emptyset, \emptyset)$ and that this equality also holds for \mathbf{mms} and \mathbf{cmms} .

2.3. Expansions.

Given a graph G and a set $F \subseteq E(G)$, we define

$$\partial_G(F) = \left(\bigcup_{e \in F} e \right) \cap \left(\bigcup_{e \in E(G) \setminus F} e \right)$$

Let G be a graph and let E_1 and E_2 be subsets of $E(G)$. An (E_1, E_2) -expansion of G is an ordering $\mathcal{E} = \langle A_1, \dots, A_r \rangle$ where

1. For $i \in \{1, \dots, r-1\}$, $E_1 \subseteq A_i \subseteq E(G) \setminus E_2$.
2. For $i \in \{1, \dots, r-1\}$, $|A_{i+1} \setminus A_i| \leq 1$.
3. $A_1 = E_1$,
4. $A_r = E(G) \setminus E_2$.

An (E_1, E_2) -expansion of G is *connected* if the following condition holds:

5. For $i \in \{1, \dots, r\}$, $G[A_i]$ is connected.

An (E_1, E_2) -expansion of G is *monotone* if the following condition holds:

6. $A_1 \subseteq \dots \subseteq A_r$.

Let $i \in \{1, \dots, r-1\}$. The *cost* of an expansion \mathcal{E} at position i is defined as $\text{cost}_G(\mathcal{E}, i) = |\partial_G(A_i)| + q_i$ where q_i is equal to one if one of the following holds

- $|A_i| \geq 2$ and $A_i \setminus A_{i-1}$ contains a pendant edge of G
- A_i consists of only one edge that is an isolated edge of G .

If none of the above two conditions hold then q_i is equal to 0. The *cost* of the expansion \mathcal{E} , denoted as $\text{cost}_G(\mathcal{E})$, is the maximum cost of \mathcal{E} among all positions $i \in \{1, \dots, r-1\}$.

We define $\mathbf{p}(G, S^{\text{in}}, S^{\text{out}})$ as the minimum cost that an $(E^{\text{in}}, E^{\text{out}})$ -expansion of $\mathbf{enh}(G, S^{\text{in}}, S^{\text{out}})$ may have, where $E^{\text{in}}, E^{\text{out}}$ are the edge extensions of $\mathbf{enh}(G, S^{\text{in}}, S^{\text{out}})$. We also define $\mathbf{mp}(G, S^{\text{in}}, S^{\text{out}})$ (if we consider only monotone $(E^{\text{in}}, E^{\text{out}})$ -expansions) and $\mathbf{cmp}(G, S^{\text{in}}, S^{\text{out}})$ (if we consider connected monotone $(E^{\text{in}}, E^{\text{out}})$ -expansions). We finally define $\mathbf{cmp}(G) = \mathbf{cmp}(G, \emptyset, \emptyset)$.

Lemma 1. *Let $(G, S^{\text{in}}, S^{\text{out}})$ be a rooted graph and let $S_1^{\text{in}} \subseteq S^{\text{in}}$ and $S_1^{\text{out}} \subseteq S^{\text{out}}$, where $G[S^{\text{in}}]$ is a connected subgraph of G . Then $\mathbf{cmp}(G, S_1^{\text{in}}, S_1^{\text{out}}) \leq \mathbf{cmp}(G, S^{\text{in}}, S^{\text{out}})$.*

Proof. Let $E^{\text{in}}, E^{\text{out}}$ and $E_1^{\text{in}}, E_1^{\text{out}}$ be the edge extensions of $\mathbf{enh}(G, S^{\text{in}}, S^{\text{out}})$ and $\mathbf{enh}(G, S_1^{\text{in}}, S_1^{\text{out}})$ respectively. Notice that, as $S_1^{\text{in}} \subseteq S^{\text{in}}$ and $S_1^{\text{out}} \subseteq S^{\text{out}}$, $E_1^{\text{in}} \subseteq E^{\text{in}}$ and $E_1^{\text{out}} \subseteq E^{\text{out}}$.

Let $\mathcal{E} = \langle A_1, \dots, A_r \rangle$ be an monotone and connected $(E^{\text{in}}, E^{\text{out}})$ -expansion of $\mathbf{enh}(G, S^{\text{in}}, S^{\text{out}})$, with cost at most k .

As $G[S^{\text{in}}]$ is a connected subgraph of G , for every vertex of $S^{\text{in}} \setminus S_1^{\text{in}}$ there exist a path connecting it with a vertex of S_1^{in} that only uses vertices of S^{in} . We define the following edge sets:

- E_1^1 contains all edges that have a vertex of S_1^{in} and a vertex of $S^{\text{in}} \setminus S_1^{\text{in}}$ as endpoints. Let $V_1 = (\bigcup_{e \in E_1^1} e) \setminus S_1^{\text{in}}$, then E_1^2 contains all edges that have both endpoints in V_1 .
- E_j^1 contains all edges that have a vertex of V_{j-1} and a vertex of $S = S^{\text{in}} \setminus (S_1^{\text{in}} \cup (\bigcup_{l=1, \dots, j-1} V_l))$ as endpoints. Let $V_j = (\bigcup_{e \in E_j^1} e) \setminus S$, then E_j^2 contains all edges that have both endpoints in V_j .

For each edge set E_j^i , $1 \leq j \leq d$ and $i \in \{1, 2\}$, where d is the maximum distance between a vertex of $S^{\text{in}} \setminus S_1^{\text{in}}$ to some vertex in S_1^{in} , we define arbitrarily an edge ordering L_j^i . We then define an ordering \mathcal{E}_1 of edge sets as follows:

- $A_1' = (A_1 \setminus E^{\text{in}}) \cup E_1^{\text{in}}$

- $A'_{1+l} = A'_{1+l-1} \cup \hat{A}_l$ for $l = 1, \dots, |E_1^1|$, where $\langle \hat{A}_1, \dots, \hat{A}_{|E_1^1|} \rangle = \mathbf{prefsec}(L_1^1)$
- $A'_{1+|E_1^1|+l} = A'_{1+|E_1^1|+l-1} \cup \hat{A}_l$ for $l = 1, \dots, |E_1^2|$, where $\langle \hat{A}_1, \dots, \hat{A}_{|E_1^2|} \rangle = \mathbf{prefsec}(L_1^2)$
- $A'_{1+|E_1^1|+|E_1^2|+\dots+|E_{j-1}^1|+|E_{j-1}^2|+l} = A'_{1+|E_1^1|+|E_1^2|+\dots+|E_{j-1}^1|+|E_{j-1}^2|+l-1} \cup \hat{A}_l$ for $l = 1, \dots, |E_j^1|$, where $\langle \hat{A}_1, \dots, \hat{A}_{|E_j^1|} \rangle = \mathbf{prefsec}(L_j^1)$
- $A'_{1+|E_1^1|+|E_1^2|+\dots+|E_{j-1}^1|+|E_{j-1}^2|+|E_j^1|+l} = A'_{1+|E_1^1|+|E_1^2|+\dots+|E_{j-1}^1|+|E_{j-1}^2|+|E_j^1|+l-1} \cup \hat{A}_l$ for $l = 1, \dots, |E_j^2|$, where $\langle \hat{A}_1, \dots, \hat{A}_{|E_j^2|} \rangle = \mathbf{prefsec}(L_j^2)$

Let $s = |E_1^1| + |E_1^2| + \dots + |E_d^1| + |E_d^2|$. Notice that there exist a $l_0 \in \{2, \dots, r\}$ such that $A'_{1+s} = (A_{l_0} \setminus E^{\text{in}}) \cup E_1^{\text{in}}$. We define a second ordering \mathcal{E}_2 of edge sets as follows: $A'_{1+s+l} = (A_{l_0+l} \setminus E^{\text{in}}) \cup E_1^{\text{in}}$ for $l = 1, \dots, r - l_0$.

Clearly $\mathcal{E}' = \mathcal{E}_1 \oplus \mathcal{E}_2$ satisfies conditions 1–4 and therefore is an $(E_1^{\text{in}}, E_1^{\text{out}})$ -expansion of $\mathbf{enh}(G, S_1^{\text{in}}, S_1^{\text{out}})$. Moreover, the monotonicity and connectivity of \mathcal{E}' follows from the monotonicity and connectivity of \mathcal{E} .

Notice that, for every $i \in \{1, \dots, r\}$, $\partial_G(A_i) = \partial_G((A_i \setminus E^{\text{in}}) \cup E_1^{\text{in}})$ therefore $\mathbf{cost}_G(\mathcal{E}', i) \leq \mathbf{cost}_G(\mathcal{E}, i)$. From this we conclude that \mathcal{E}' has cost at most k . \square

Let $\mathbf{G}_1, \dots, \mathbf{G}_r$ be rooted graphs such that $\mathbf{G}_i = (G_i, S_i^{\text{in}}, S_i^{\text{out}})$ where $V(G_i) \cap V(G_{i+1}) = S_i^{\text{out}} = S_{i+1}^{\text{in}}$, $i \in \{1, \dots, r-1\}$. We define, $\mathbf{glue}(\mathbf{G}_1, \dots, \mathbf{G}_r) = (G_1 \cup \dots \cup G_r, S_1^{\text{in}}, S_r^{\text{out}})$.

Lemma 2. *Let $\mathbf{G}_1, \dots, \mathbf{G}_r$ be rooted graphs such that $\mathbf{G}_i = (G_i, S_i^{\text{in}}, S_i^{\text{out}})$ where $V(G_i) \cap V(G_{i+1}) = S_i^{\text{out}} = S_{i+1}^{\text{in}}$, $i \in \{1, \dots, r-1\}$. Then*

$$\mathbf{cmp}(\mathbf{glue}(\mathbf{G}_1, \dots, \mathbf{G}_r)) \leq \max\{\mathbf{cmp}(\mathbf{G}_i) \mid i \in \{1, \dots, r\}\}.$$

Proof. Let $E_i^{\text{in}}, E_i^{\text{out}}$ be the edge extensions and $\mathcal{E}_i = \langle A_1^i, \dots, A_{l_i}^i \rangle$ be an monotone and connected $(E_i^{\text{in}}, E_i^{\text{out}})$ -expansion of $\mathbf{enh}(G_i, S_i^{\text{in}}, S_i^{\text{out}})$, for every $i \in \{1, \dots, r\}$. Clearly $\mathcal{E} = \langle A_1^1, \dots, A_{l_1}^1, A_{l_1}^1 \cup A_2^2, \dots, A_{l_1}^1 \cup A_{l_2}^2, \dots, (\cup_{1 \leq i < r} A_{l_i}^i) \cup A_2^r, \dots, (\cup_{1 \leq i < r} A_{l_i}^i) \cup A_{l_r}^r \rangle$ is an $(E_1^{\text{in}}, E_r^{\text{out}})$ -expansion of $\mathbf{glue}(\mathbf{G}_1, \dots, \mathbf{G}_r)$ and, as expansions \mathcal{E}_i , $i \in \{1, \dots, r\}$ are monotone and connected, conditions 5 and 6 hold.

We observe that $\mathbf{cost}_{G_1 \cup \dots \cup G_r}(\mathcal{E}) \leq \max\{\mathbf{cost}_{G_1}(\mathcal{E}_1), \dots, \mathbf{cost}_{G_r}(\mathcal{E}_r)\}$, therefore $\mathbf{cmp}(\mathbf{glue}(\mathbf{G}_1, \dots, \mathbf{G}_r)) \leq \max\{\mathbf{cmp}(\mathbf{G}_i) \mid i \in \{1, \dots, r\}\}$. \square

Lemma 3. *For every graph G , $\mathbf{cmms}(G, S^{\text{in}}, S^{\text{out}}) = \mathbf{cmp}(G, S^{\text{in}}, S^{\text{out}})$.*

Proof. Assume that $G^* = \mathbf{enh}(G, S^{\text{in}}, S^{\text{out}})$ has a complete search strategy \mathcal{S} satisfying conditions (i) – (iii) with cost at most k . We construct an edge ordering of $E(G)$ as follows. Observe that, because of the monotonicity of \mathcal{S} , $E^{(i)} = E(\mathcal{S}, i) \setminus E(\mathcal{S}, i-1)$. For every $i \in \{1, \dots, |\mathcal{S}|\}$, we define L_i by taking any ordering of the set $E^{(i)}$ and insisting that, if $E^{(i)}$ contains some sliding edge,

this edge will be the first edge of L_i . Let $\mathcal{E} = \langle A_0, \dots, A_r \rangle$ be the sequence of prefixes of $L_1 \oplus \dots \oplus L_{|\mathcal{S}|}$, including the empty set (that is $A_0 = \emptyset$). Notice that, because of Condition (i), $A_s = E^{\text{in}}$ for some $s \in \{1, \dots, |\mathcal{S}|\}$, and, because of Condition (iii), $A_t = E(G) \setminus E^{\text{out}}$, for some $t \in \{1, \dots, |\mathcal{S}|\}$. We now claim that $\mathcal{E}' = \langle A_s, \dots, A_t \rangle$ is an $(E^{\text{in}}, E^{\text{out}})$ -expansion of G^* . Indeed, Condition (1) holds because of Condition (ii) and Conditions (2) – (4) hold because of the construction of \mathcal{E}' . Moreover, the connectivity and the monotonicity of \mathcal{E}' follow directly from the connectivity and the monotonicity of \mathcal{S} .

It remains to prove that the cost of \mathcal{E}' is at most k . For each $j \in \{0, \dots, |\mathcal{E}'|\}$ we define i_j such that the unique edge in $A_j \setminus A_{j-1}$ is an edge in $E^{(i_j)}$ and we define h_j such that $A_{h_j} \setminus A_{h_j-1}$ contains the first edge of L_{i_j} . Notice now that the cost of \mathcal{E}' at positions h_j to j is upper bounded by the cost of \mathcal{E}' at position h_j . Therefore, it is enough to prove that the cost of \mathcal{E}' at position h_j is at most k . Recall that this cost is equal to $|\partial_G(A_{h_j})| + q_{h_j}$. We distinguish two cases:

Case 1. If $q_{h_j} = 0$, then the cost of \mathcal{E}' at position h_j is equal to $|\partial_G(A_{h_j})|$. As \mathcal{S} is monotone, all vertices in $\partial_G(A_{h_j})$ should be occupied by searchers after the i_j -th move of \mathcal{S} and therefore the cost of \mathcal{E}' at position h_j is at most k .

Case 2. If $q_{h_j} = 1$, then the i_j -th move of \mathcal{S} is either the placement of a searcher on a pendant vertex x or the sliding of a searcher along a pendant edge $\{y, x\}$ towards its pendant vertex x . In both cases, $x \notin \partial_G(A_{h_j})$ and all vertices in $\partial_G(A_{h_j})$ should be occupied by searchers after the i_j -th move. In the first case, there are in total at least $|\partial_G(A_{h_j})| + 1$ searchers on the graph and we are done. In the second case, we observe that, because of monotonicity, $\partial_G(A_{h_j}) = \partial_G(A_{h_j-1}) \setminus \{y\}$. As after the $(h_j - 1)$ -th move all vertices of $\partial_G(A_{h_j-1})$ were occupied by searchers, we obtain that $|\partial_G(A_{h_j})| \leq k - 1$ and thus the cost of \mathcal{E}' at position h_j is at most k .

Now assume that there exist a monotone and connected $(E^{\text{in}}, E^{\text{out}})$ -expansion of G^* , say $\mathcal{E} = \langle A_1, \dots, A_r \rangle$, with cost at most k . We can additionally assume that \mathcal{E} is properly monotone; this can be done by discarding additional repetitions of a set in \mathcal{E} .

Moreover, starting from \mathcal{E} , we can construct a monotone and connected $(E^{\text{in}}, E^{\text{out}})$ -expansion of G^* , with cost at most k , say $\mathcal{E}' = \langle A'_1, \dots, A'_r \rangle$, with the following additional property:

Expansion property: For every $i \in \{1, \dots, r - 1\}$ for which $V(A'_i) \subset V(A'_{i+1})$, A'_i contains all edges of G^* with both endpoints in $V(A'_i)$.

This can be accomplished by a series of appliances of the following rule:

Rule: Let $V(A_i) \subset V(A_{i+1})$ for some i and let $L = \langle e_1, \dots, e_n \rangle$ be an ordering of the edges $E(G^*) \setminus A_i$ with both endpoints in $V(A_i)$. For every $j \leq i$ define $A'_j = A_j$. Then, define $A'_{i+1} = A_i \cup \{e_1\}$, $A'_{i+2} = A_i \cup \{e_1, e_2\}$ and so on until $A'_{i+n} = A_i \cup \{e_1, \dots, e_n\}$. Finally, for every $j \geq i + n$, define

$$A'_j = A_j \cup \{e_1, \dots, e_n\}.$$

One can easily check that, after every application of this Rule, the constructed sequence of edge sets is indeed an $(E^{\text{in}}, E^{\text{out}})$ -expansion of G^* and furthermore it is monotone and connected. Notice that, for $j = 1, \dots, n$, $\partial_{G^*}(A'_{i+j}) \subseteq \partial_{G^*}(A_i)$ and for $j \geq i + n$, $|\partial_{G^*}(A'_j)| \leq |\partial_{G^*}(A_j)|$. Moreover, if $|A_i| \geq 2$ and $A_i \setminus A_{i-1}$ contains a pendant edge of G^* then for every $j \in \{1, \dots, n\}$, $|A'_{i+j}| \geq 2$ and $A'_{i+j} \setminus A'_{i+j-1}$ contains the same pendant edge of G^* , hence the cost of \mathcal{E}' is at most k (notice that if A_i consists of only one edge that is an isolated edge of G^* then there does not exist an edge in $E(G^*) \setminus A_i$ with both endpoints in $V(A_i)$, therefore we do not need to apply this rule).

For the rest of the proof, we will consider that the Expansion property holds for the given $(E^{\text{in}}, E^{\text{out}})$ -expansion of G^* .

Our target is to define a $(S^{\text{in}}, S^{\text{out}})$ -complete monotone search strategy \mathcal{S} of G^* with cost at most k .

The first $|S^{\text{in}}|$ moves of \mathcal{S} will be $\mathbf{p}(u^{\text{in}})$ and the next $|S^{\text{in}}|$ will be $\mathbf{s}(u^{\text{in}}, v_i^{\text{in}})$. We denote this sequence of moves by \mathcal{S}_0 . Notice that $E(\mathcal{S}, 2|S^{\text{in}}|) = A_1$.

For every vertex u in the set $V^* = V(G^*) \setminus S^{\text{in}} \setminus \{u^{\text{out}}\}$, we define l_u to be the first integer in $\{1, \dots, r\}$ such that $u \in V(A_{l_u})$.

Let $L = \langle u_1, \dots, u_{|V^*|} \rangle$ be an ordering of V^* such that $i \leq j$ when $l_{u_i} \leq l_{u_j}$. Notice that, for each $i \in \{1, \dots, |V^*|\}$, the vertex u_i is an endpoint of the unique edge e_i in $A_{l_{u_i}-1} \setminus A_{l_{u_i}}$ and let v_i be the other endpoint of e_i . Notice that, because of the connectivity and the monotonicity of \mathcal{E} , $v_i \in \partial_{G^*}(A_{l_{u_i}-1})$. We also observe that u_i is pendant iff $u_i \notin \partial_{G^*}(A_{l_{u_i}})$. We define $E' = \{e_1, \dots, e_{|V^*|}\}$ and we call a set A_j , $j \in \{1, \dots, r\}$, crucial iff $|A_{j-1} \cap E'| < |A_j \cap E'|$.

For each $i \in \{1, \dots, |V^*|\}$, we define a sequence \mathcal{S}_i of moves as follows: If $v_i \in \partial_{G^*}(A_{l_{u_i}})$ then the first move of \mathcal{S}_i is $\mathbf{p}(u_i)$, otherwise it is $\mathbf{s}(v_i, u_i)$. The rest of the moves in \mathcal{S}_i are the removals, one by one, of the searchers in $\partial_{G^*}(A_{l_{u_i}-1}) \setminus \partial_{G^*}(A_{l_{u_i}})$. Then we define $\mathcal{S} = \mathcal{S}_0 \oplus \mathcal{S}_1 \oplus \dots \oplus \mathcal{S}_{|V^*|}$.

Notice that, according to the Expansion property, all edges of the sets A_j , for $j = 1, \dots, l_{u_1}$ have both endpoints in S^{in} . Moreover, for every $i \in \{1, \dots, |V^*| - 1\}$ all edges of the sets A_j , for $j = l_{u_i}, \dots, l_{u_{i+1}} - 1$, have both endpoints in $V(A_{l_{u_i}})$ and all edges of the sets A_j , for $j = l_{u_{|V^*|}}, \dots, r$, have both endpoints in $V(A_{l_{u_{|V^*|}}})$.

First we show that the following claim is true:

Claim 1. For every A_j , $j \in \{1, \dots, r\}$, the vertices of $\partial_{G^*}(A_j)$ are exactly the vertices occupied by searchers after the last move of \mathcal{S}_{m_j} , where m_j is the index of the edge in $(A \cap E') \setminus (A_{j-1} \cap E')$, where A is the first crucial set of \mathcal{E} such that $A_j \subseteq A$.

Clearly, this is true for $A_1 = E^{\text{in}}$. Assume that it holds for $A_{j'}$.

We will show that the vertices in $\partial_{G^*}(A_{j'+1})$ are exactly the vertices occupied by searchers after the last move of $\mathcal{S}_{m_{j'+1}}$.

If $A_{j'+1}$ is not crucial then $\partial_{G^*}(A_{j'+1}) \subseteq \partial_{G^*}(A_{j'})$ and $m_{j'+1} = m_{j'}$, there-

fore Claim 1 holds.

Now, if $A_{j'+1}$ is crucial and $\{e_{m_{j'+1}}\} = (A_{j'+1} \cap E') \setminus (A_{j'} \cap E')$, then $v_{m_{j'+1}} \in \partial_{G^*}(A_{j'})$ and therefore must be occupied by a searcher. We distinguish three cases:

Case 1. If $v_{m_{j'+1}} \in \partial_{G^*}(A_{j'+1})$ and $u_{m_{j'+1}} \in \partial_{G^*}(A_{j'+1})$, then $\partial_{G^*}(A_{j'+1}) = \partial_{G^*}(A_{j'}) \cup \{u_{m_{j'+1}}\}$ and the first move in $\mathcal{S}_{m_{j'+1}}$ will be $p(u_{m_{j'+1}})$.

Case 2. If $v_{m_{j'+1}} \in \partial_{G^*}(A_{j'+1})$ and $u_{m_{j'+1}} \notin \partial_{G^*}(A_{j'+1})$ then $\partial_{G^*}(A_{j'+1}) = \partial_{G^*}(A_{j'})$.

Case 3. If $v_{m_{j'+1}} \notin \partial_{G^*}(A_{j'+1})$, then $\partial_{G^*}(A_{j'+1}) = (\partial_{G^*}(A_{j'}) \setminus \{v_{m_{j'+1}}\}) \cup \{u_{m_{j'+1}}\}$, and the first move in $\mathcal{S}_{m_{j'+1}}$ will be $s(v_{m_{j'+1}}, u_{m_{j'+1}})$.

Observe that in all three cases the Claim 1 holds.

Let $V_{\mathcal{S}}(i)$ be the set of vertices already visited by searchers after the i -th move of \mathcal{S} , and let $V_{\mathcal{S}} = \langle V_{\mathcal{S}}(1), \dots, V_{\mathcal{S}}(r) \rangle$. Notice that this sequence is monotone and that if the i -th move belong to the subsequence \mathcal{S}_j , then $V_{\mathcal{S}}(i) = V(A_{l_{u_j}})$. We must next prove the following claim:

Claim 2: For every $i \in \{1, \dots, |\mathcal{S}|\}$, all edges of $G^*[V_{\mathcal{S}}(i)]$ are clean.

Clearly, the claim is true for $i \in \{1, \dots, 2 \cdot |S^{\text{in}}|\}$. Assume that it holds for some $i \in 2 \cdot |S^{\text{in}}| + 1, \dots, r$, we will show that all edges of $G^*[V_{\mathcal{S}}(i+1)]$ are clean. We must distinguish three cases about the $(i+1)$ -th move:

Case 1. It is a removal, say $r(u)$. Notice that $G^*[V_{\mathcal{S}}(i+1)] = G^*[V_{\mathcal{S}}(i)]$, therefore the Claim will not be true if $r(u)$ is a recontamination move. In this case, there exist an edge connecting u with a vertex not in $V_{\mathcal{S}}(i)$, say v . As $u \in \partial_{G^*}(A_{l_{u_j}-1}) \setminus \partial_{G^*}(A_{l_{u_j}})$, for some $j \in \{1, \dots, |V^*|\}$, all edges with u as endpoint must belong to $A_{l_{u_j}}$, therefore $\{u, v\} \in A_{l_{u_j}}$. But $V_{\mathcal{S}}(i) = V(A_{l_{u_j}})$, a contradiction.

Case 2. It is a placement of searcher say $p(u)$. By the definition of \mathcal{S} , there exist an edge $\{u, v\}$, where v is a vertex in $V_{\mathcal{S}}(i)$. Notice that, according to our search game, all such edges are clean after $p(u)$, thus all edges of $G^*[V_{\mathcal{S}}(i+1)]$ are clean.

Case 3. It is a slide, say $s(v_j, u_j)$, for some $j \in \{1, \dots, |V^*|\}$. As in the previous case, $G^*[V_{\mathcal{S}}(i+1)]$ contains all edges of $G^*[V_{\mathcal{S}}(i)]$ and additional all edges with u_j as the first endpoint and a vertex $v \in V_{\mathcal{S}}(i)$ as the other. According to our search game, after the i -th move there must be searcher in v_j , therefore due to Claim 1, $v_j \in \partial_{G^*}(A_{l_{u_j}})$. Notice that, the Claim will not be true if $s(v_j, u_j)$ is a recontamination move, i.e., there exist an edge connecting v_j with a vertex, say u , not in $V_{\mathcal{S}}(i) = V(A_{l_{u_j}})$. As $v_j \notin \partial_{G^*}(A_{l_{u_j}})$, all edges with u as endpoint

must belong to $A_{l_{u_j}}$, therefore $\{v_j, u\} \in A_{l_{u_j}}$, a contradiction.

In all three cases we show that after the $(i+1)$ -th move of S all edges of $G^*[V_S(i+1)]$ are clean, therefore Claim 2 is true.

We will now prove that \mathcal{S} is a (S_1, S_2) -complete strategy for G^* . Clearly, Condition (i) holds for every strategy starting with \mathcal{S}_0 . Moreover, Condition (ii) holds as v^{out} is not a vertex of V^* and therefore, no placement on u^{out} or sliding towards u^{out} appears in \mathcal{S} . Notice that, according to Claim 2, for every $i \in \{1, \dots, |V^*| - 1\}$, $E(\mathcal{S}, |\mathcal{S}_0 \oplus \dots \oplus \mathcal{S}_{i-1}| + 1) = \dots = E(\mathcal{S}, |\mathcal{S}_0 \oplus \dots \oplus \mathcal{S}_{i-1}| + |\mathcal{S}_i|) = A_{l_{u_{i+1}} - 1}$, and that for $i = |V^*|$, $E(\mathcal{S}, |\mathcal{S}_0 \oplus \dots \oplus \mathcal{S}_{|V^*|}|) = A_r$, therefore Condition (iii) holds.

By the definition of \mathcal{S} , it is clear that \mathcal{S} is a connected search strategy, moreover, according to Claim 2, \mathcal{S} is monotone. It remains to prove that \mathcal{S} has cost at most k . For the first $2|S^{\text{in}}|$ moves, we use $|S^{\text{in}}| = \text{cost}_{G^*}(\mathcal{E}, 1) \leq k$ searchers. Assume that after j moves exactly k searchers are occupying vertices of G^* and that the $(j+1)$ -th move is $\mathbf{p}(u_i)$, for some $i \in \{i, \dots, |V^*|\}$. Then the vertices in $\partial_{G^*}(A_{l_{u_i} - 1})$ are exactly the vertices occupied by the k searchers, therefore $|\partial_{G^*}(A_{l_{u_i} - 1})| = k$. Observe that, if u_i is not pendant, then $\partial_{G^*}(A_{l_{u_i}}) = \partial_{G^*}(A_{l_{u_i} - 1}) \cup \{u_i\}$, therefore $|\partial_{G^*}(A_{l_{u_i}})| = k + 1$, a contradiction and if u_i is pendant then $\partial_{G^*}(A_{l_{u_i}}) = \partial_{G^*}(A_{l_{u_i} - 1})$ and the cost of \mathcal{E} at position l_{u_i} is $|\partial_{G^*}(A_{l_{u_i}})| + 1 = k + 1$, again a contradiction. Thus, for every move of \mathcal{S} at most k searchers are occupying vertices of G^* . \square

2.4. Contractions.

Let $(G_1, S_1^{\text{in}}, S_1^{\text{out}})$ and $(G_2, S_2^{\text{in}}, S_2^{\text{out}})$ be rooted graphs. We say that $(G_1, S_1^{\text{in}}, S_1^{\text{out}})$ is a *contraction* of $(G_2, S_2^{\text{in}}, S_2^{\text{out}})$ and we denote this fact by $(G_1, S_1^{\text{in}}, S_1^{\text{out}}) \preceq (G_2, S_2^{\text{in}}, S_2^{\text{out}})$ if there exist a surjection $\phi : V(G_2) \rightarrow V(G_1)$ such that:

1. for every vertex $v \in V(G_1)$, $G_2[\phi^{-1}(v)]$ is connected
2. for every two distinct vertices $u, v \in V(G_1)$, it holds that $\{v, u\} \in E(G_1)$ if and only if the graph $G_2[\phi^{-1}(v) \cup \phi^{-1}(u)]$ is connected
3. $\phi(S_2^{\text{in}}) = S_1^{\text{in}}$
4. $\phi(S_2^{\text{out}}) = S_1^{\text{out}}$

We also write $(G_1, S_1^{\text{in}}, S_1^{\text{out}}) \preceq_{\phi} (G_2, S_2^{\text{in}}, S_2^{\text{out}})$ to make clear the function that certifies the contraction relation. We say that G_1 is a *contraction* of G_2 if $(G_1, \emptyset, \emptyset) \preceq (G_2, \emptyset, \emptyset)$ and we denote this fact by $G_1 \preceq G_2$. If furthermore G_1 is not isomorphic to G_2 we say that G_1 is a *proper contraction* of G_2 .

We define the *minor* relation for the two rooted graph by removing in the second property the demand that if $\{u, v\} \notin E(G_1)$ then $G_2[\phi^{-1}(v) \cup \phi^{-1}(u)]$ is not connected. We denote the minor relation by $(G_1, S_1^{\text{in}}, S_1^{\text{out}}) \leq (G_2, S_2^{\text{in}}, S_2^{\text{out}})$. Again, we say that G_1 is a *minor* of G_2 if $(G_1, \emptyset, \emptyset) \leq (G_2, \emptyset, \emptyset)$ and we denote this fact by $G_1 \leq G_2$.

Lemma 4. *If $(G_1, S_1^{\text{in}}, S_1^{\text{out}})$ and $(G_2, S_2^{\text{in}}, S_2^{\text{out}})$ are rooted graphs and $(G_1, S_1^{\text{in}}, S_1^{\text{out}}) \preceq (G_2, S_2^{\text{in}}, S_2^{\text{out}})$, then $\mathbf{cmp}(G_1, S_1^{\text{in}}, S_1^{\text{out}}) \leq \mathbf{cmp}(G_2, S_2^{\text{in}}, S_2^{\text{out}})$.*

Proof. Suppose that $\mathcal{E} = \langle A_1, \dots, A_r \rangle$ is a monotone $(E_2^{\text{in}}, E_2^{\text{out}})$ -expansion of $G_2^* = \mathbf{enh}(G_2, S_2^{\text{in}}, S_2^{\text{out}})$ with cost at most k . Our target is to construct a monotone $(E_1^{\text{in}}, E_1^{\text{out}})$ -expansion of $G_1^* = \mathbf{enh}(G_1, S_1^{\text{in}}, S_1^{\text{out}})$ with cost at most k .

Let ϕ be a function where $(G_1, S_1^{\text{in}}, S_1^{\text{out}}) \preceq_\phi (G_2, S_2^{\text{in}}, S_2^{\text{out}})$. We consider an extension ψ of ϕ that additionally maps u_2^{in} to u_1^{in} and u_2^{out} to u_1^{out} . Notice that the construction of ψ yields the following:

$$(G_1^*, S_1^{\text{in}} \cup \{u_1^{\text{in}}\}, S_1^{\text{out}} \cup \{u_1^{\text{out}}\}) \preceq_\phi (G_2^*, S_2^{\text{in}} \cup \{u_2^{\text{in}}\}, S_2^{\text{out}} \cup \{u_2^{\text{out}}\})$$

Given an edge $f = \{x, y\} \in E(G_1)$ we consider the set E_f containing all edges of G_2 with one endpoint in $\psi^{-1}(x)$ and one endpoint in $\psi^{-1}(y)$. We now pick, arbitrarily, an edge in E_f and we denote it by e_f . We also set $E' = \{e_f \mid f \in E(G_1)\}$. Then it is easy to observe that $\mathcal{E}' = \langle A_1 \cap E', \dots, A_r \cap E' \rangle$ is a connected expansion of G_1^* and that the cost of \mathcal{E}' at step i is no bigger than the cost of \mathcal{E} at the same step, where $i \in \{1, \dots, r-1\}$. \square

Lemma 5. *If G_1 and G_2 are two graphs and $G_1 \preceq G_2$, then $\mathbf{cs}(G_1) \leq \mathbf{cs}(G_2)$.*

Proof. First observe that if this is the case any contraction of G_2 can be derived by applying a finite number of edge-contractions of some edges in $E(G_2)$.

It suffices to prove that the Lemma hold if G_1 is obtained by the contraction of edge $e = \{u, v\} \in E(G_1)$ to vertex x_{uv} . Let \mathcal{S} be a connected search strategy for G_2 that in any step uses at most k searchers. Based on \mathcal{S} we will construct a search strategy \mathcal{S}' for G_1 . Let i be an integer in $\{1, \dots, |\mathcal{S}|\}$. We distinguish eight cases:

Case 1: If the i -th move of \mathcal{S} is $\mathbf{p}(x)$ for some vertex $x \notin \{u, v\}$ then the next move of \mathcal{S}' will be $\mathbf{p}(x)$.

Case 2: If the i -th move of \mathcal{S} is $\mathbf{r}(x)$ for some vertex $x \notin \{u, v\}$ then the next move of \mathcal{S}' will be $\mathbf{r}(x)$.

Case 3: If the i -th move of \mathcal{S} is $\mathbf{s}(x, y)$ for some vertices $x, y \notin \{u, v\}$ then the next move of \mathcal{S}' will be $\mathbf{s}(x, y)$.

Case 4: If the i -th move of \mathcal{S} is $\mathbf{p}(u)$ or $\mathbf{p}(v)$ then the next move of \mathcal{S}' will be $\mathbf{p}(x_{uv})$.

Case 5: If the i -th move of \mathcal{S} is $\mathbf{r}(u)$ or $\mathbf{r}(v)$ then the next move of \mathcal{S}' will be $\mathbf{r}(x_{uv})$.

Case 6: If the i -th move of \mathcal{S} is $\mathbf{s}(z, u)$ or $\mathbf{p}(z, v)$ for some vertex z then the next move of \mathcal{S}' will be $\mathbf{s}(z, x_{uv})$.

Case 7: If the i -th move of \mathcal{S} is $\mathbf{s}(u, z)$ or $\mathbf{p}(v, z)$ for some vertex z then the next move of \mathcal{S}' will be $\mathbf{s}(x_{uv}, z)$.

Case 8: If the i -th move of \mathcal{S} is $\mathbf{s}(u, v)$ or $\mathbf{p}(v, u)$ then the next move of \mathcal{S}' will be defined according the lateral cases from the $(i + 1)$ -th move of \mathcal{S} .

Observe that \mathcal{S}' is a complete search strategy for G_1 . Furthermore, as \mathcal{S} is connected, \mathcal{S}' must also be connected. Finally, it is clear that \mathcal{S}' at any step uses at most k searchers, thus $\mathbf{cms}(G_1) \leq k$. \square

2.5. Parameters and obstructions.

We denote by \mathcal{G} the class of all graphs. A *graph parameter* is a function $f : \mathcal{G} \rightarrow \mathbb{N}$. Given a graph parameter f and an integer $k \in \mathbb{N}$ we define the *graph class* $\mathcal{G}[f, k]$, containing all the graphs $G \in \mathcal{G}$ where $f(G) \leq k$.

Let \mathcal{H} be a graph class. We denote by $\mathbf{obs}(\mathcal{H})$ the set of all graphs in $\mathcal{G} \setminus \mathcal{H}$ that are minimal with respect to the relation \preceq .

2.6. Cut-vertices and blocks.

We call the 2-connected components of a graph G *blocks*. If the removal of an edge in a graph increases the number of its connected components then it is called *bridge*. We consider the subgraph of G induced by the endpoints of a bridge of G as one of its blocks and we call it *trivial block* of G .

A *cut-vertex* of a graph G is a vertex such that $G \setminus x$ has more connected components than G . Given a graph G a *cut-vertex of a block* B of G is a cut-vertex of G that belongs in $V(B)$.

Let G be a graph and let $x \in V(G)$. We define

$$\mathcal{C}_G(x) = \{(x, G[V(C) \cup \{x\}]) \mid C \text{ is a connected component of } G \setminus x\}.$$

Let B be a block of G and let x be a cut-vertex of B . We denote by $\mathcal{C}_G(x, B)$ the (unique) graph in $\mathcal{C}_G(x)$ that contains B as a subgraph and by $\bar{\mathcal{C}}_G(x, B)$ the graphs in $\mathcal{C}_G(x)$ that do not contain B .

2.7. Outerplanar graphs.

We call a graph G *outerplanar* if it can be embedded in the plane such that all its vertices are incident to its infinite face (also called *outer face*). This embedding, when exists, is unique up to homeomorphism and, from now on, each outerplanar graph is accompanied with such an embedding. An edge $e \in E(G)$ is called *outer edge* of G , if it is incident to the outer face of G , otherwise is called a *chord* of G .

A face F of an outer planar graph that is different than the outer face, is called *haploid* if and only if at most one edge incident to F is a chord, otherwise F is a *inner* face. A vertex $u \in V(G)$ is *haploid* if it is incident to an haploid face and *inner* if it is incident to an inner face (notice that some vertices can be both inner and haploid). A vertex of G that is not inner or haploid is called *outer*. We call a chord *haploid* if it is incident to an haploid face. Non-haploid chords are called *internal chords*.

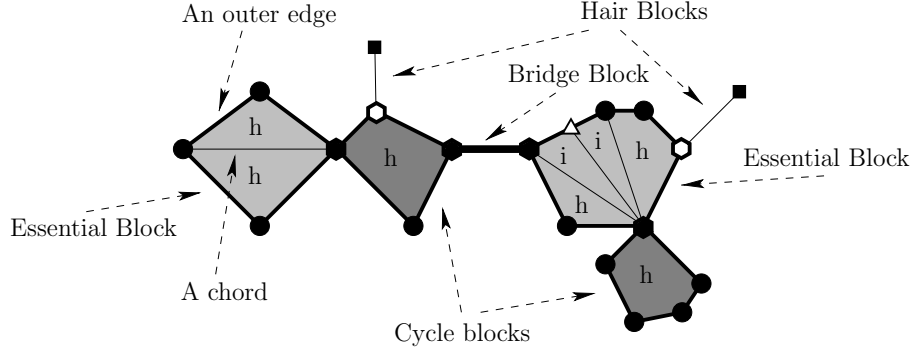


Figure 1: A outerplanar graph and its blocks. The cut-vertices are hexagonal and the outer vertices are squares. Inner and haploid faces are denoted by “i” and “h” respectively. There are, in total, four inner vertices (all belonging to the essential block on the right) and, among them only the triangular one is not an haploid vertex. The white hexagonal vertices are the light cut-vertices while the rest of the hexagonal vertices are the heavy ones.

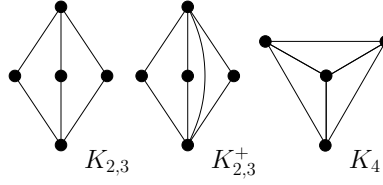


Figure 2: The set \mathcal{O}_1 .

Observation 1. *A block of a connected outerplanar graph with more than one edge can be one of the following.*

- a hair block: it is a trivial block containing exactly one vertex of degree 1 in G .
- a bridge block: it is a trivial block that is not a hair-block.
- a cycle block: if it is a chordless non-trivial block, or
- an essential block: if it is a non-trivial block with at least one chord.

Let G be a connected outerplanar graph with more than one edges. Given a cut-vertex c of G , we say that c is *light* if it is the (unique) cut-vertex of exactly one hair block. If a cut-vertex of G is not light then it is *heavy*.

It is known that the class of outerplanar graphs is closed under the relations \leq, \preceq and that a graph is outerplanar if and only if $K_4 \not\leq G$ and $K_{2,3} \not\leq G$.

Let $K_{2,3}^+$ be the graph obtained by $K_{2,3}$ after connecting the two vertices of degree 3 (Figure 2).

Lemma 6. *If \mathcal{H} is the class of all outerplanar graphs, then $\mathbf{obs}(\mathcal{H}) = \mathcal{O}_1$.*

Proof. Observe that the graphs in \mathcal{O}_1 cannot be embedded in the plane in such a way that all of its vertices are incident to a single face and therefore neither the graphs in \mathcal{O}_1 , neither the graphs that contain as a contraction a graph in \mathcal{O}_1 , can be outerplanar.

To complete the proof, one must show that every non-outerplanar graph can be contracted to a graph in \mathcal{O}_1 . Let G be non-outerplanar, then $K_4 \leq G$ or $K_{2,3} \leq G$. Clearly, as K_4 is a clique, $K_4 \leq G$ implies that $K_4 \preceq G$. Suppose now that $K_{2,3} \leq G$. Let V_x, V_y, V_1, V_2, V_3 be the vertex sets of the connected subgraphs of G that are contracted towards creating the vertices of $K_{2,3}$ (V_x and V_y are contracted to vertices of degree 3). If there is no edge in G between two vertices in V_a and V_b for some $(a, b) \in \{(x, y), (1, 2), (2, 3), (1, 3)\}$ then $K_{2,3} \preceq G$. If the only such edge is between V_x and V_y then $K_{2,3}^+ \preceq G$ and in any other case, $K_4 \preceq G$. \square

3. Obstructions for Graphs with cmms at most 2

In this section we give the obstruction set for graphs with connected monotone mixed search number at most 2 and we prove its correctness.

3.1. The obstruction set for $k = 2$

Let $\mathcal{D}^1 = \mathcal{O}_1 \cup \dots \cup \mathcal{O}_{12}$ where \mathcal{O}_1 is depicted in Figure 2, $\mathcal{O}_2, \dots, \mathcal{O}_9$ are depicted in Figure 3 and \mathcal{O}_{10} and \mathcal{O}_{11} and \mathcal{O}_{12} are constructed as follows.

\mathcal{O}_{10} : contains every graph that can be constructed by taking three disjoint copies of some graphs in Figure 8 and then identify the vertices denoted by v in each of them to a single vertex. There are, in total, 35 graphs generated in this way.

\mathcal{O}_{11} : contains every graph that can be constructed by taking two disjoint copies of some graphs in Figure 12 and then identify the vertices denoted by v in each of them to a single vertex. There are, in total, 78 graphs generated in this way.

\mathcal{O}_{12} : contains every graph that can be constructed by taking two disjoint copies of some graphs in Figure 13 and then identify the vertices denoted by v in each of them to a single vertex. There are, in total, 21 graphs generated in this way.

Observe that, \mathcal{D}^1 contains 177 graphs.

3.2. Proof strategy

Lemma 7. $\mathcal{D}^1 \subseteq \mathbf{obs}(\mathcal{G}[\mathbf{cmp}, 2])$.

Proof. From Lemma 4, it is enough to check that for every $G \in \mathcal{D}^1$, the following two conditions are satisfied (i) $\mathbf{cmp}(G) \geq 3$ and (ii) for every edge e of G it holds that $\mathbf{cmp}(G/e) \leq 2$. One can verify that this is correct by inspection, as this concerns only a finite amount of graphs and, for each of them, there exists a finite number of edges to contract. \square

Lemma 8. $\mathcal{D}^1 \supseteq \mathbf{obs}(\mathcal{G}[\mathbf{cmp}, 2])$.

The rest of this section is devoted to the proof of Lemma 8. For this, our strategy is to consider the set

$$\mathcal{Q} = \mathbf{obs}(\mathcal{G}[\mathbf{cmp}, 2]) \setminus \mathcal{D}^1$$

and prove that $\mathcal{Q} = \emptyset$ (Lemma 21). For this, we need a series of structural results whose proofs use the following three fundamental properties of the set \mathcal{Q} .

Lemma 9. *Let $G \in \mathcal{Q}$. Then the following hold.*

- i. $\mathbf{cmp}(G) \geq 3$.
- ii. *If H is a proper contraction of G , then $\mathbf{cmp}(H) \leq 2$.*
- iii. *G does not contain any of the graphs in \mathcal{D}^1 as a contraction.*

Proof. Properties i. and ii. hold because $G \in \mathbf{obs}(\mathcal{G}[\mathbf{cmp}, 2])$. For property iii. suppose, to the contrary, that G contains some graph in $H \in \mathcal{D}^1$ as a contraction. From Lemma 7, $H \in \mathbf{obs}(\mathcal{G}[\mathbf{cmp}, 2])$. Clearly, H is different than G as \mathcal{Q} does not contain members of \mathcal{D}^1 . Therefore, H is a proper contraction of G and, from property ii., $\mathbf{cmp}(H) \leq 2$. This contradicts to the fact that $H \in \mathbf{obs}(\mathcal{G}[\mathbf{cmp}, 2])$ and thus $\mathbf{cmp}(H) \geq 3$. \square

3.3. Basic structural properties

Lemma 10. *Let $G \in \mathcal{Q}$. The following hold:*

- 1. *G is outerplanar.*
- 2. *Every light cut-vertex of G has degree at least 3.*
- 3. *Every essential block B of G , has exactly two haploid faces.*
- 4. *Every block of G , has at most 3 cut-vertices*
- 5. *Every cut-vertex of a non-trivial block of G is an haploid vertex.*
- 6. *Every block of G contains at most 2 heavy cut-vertices.*
- 7. *If a block of G has 3 cut-vertices, then there are two, say x and y , of these vertices that are not both heavy and are connected by an haploid edge.*

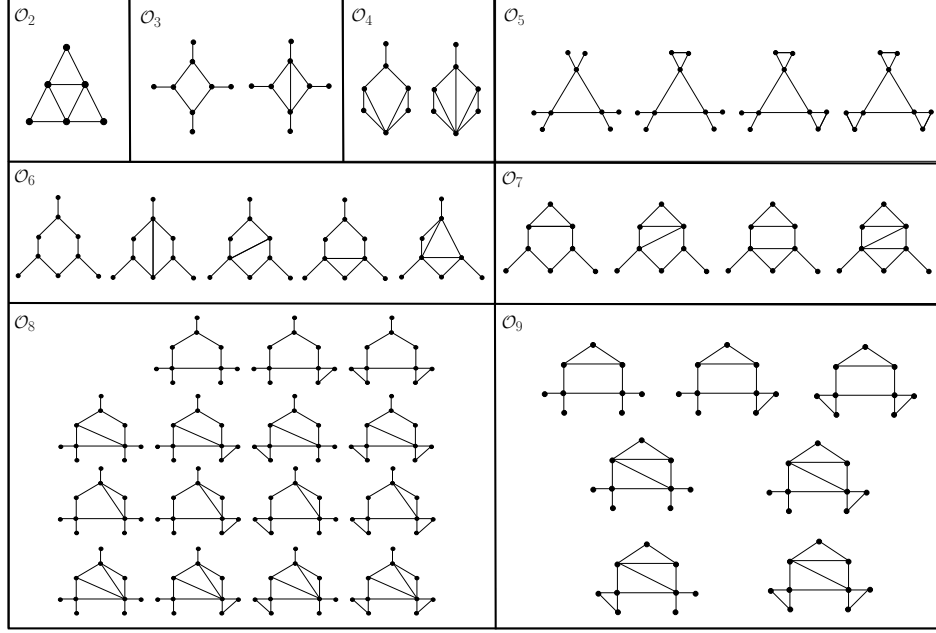


Figure 3: The sets of graphs in \mathcal{D}_1 .

8. If an essential block of G with haploid faces F_1 and F_2 has two heavy cut-vertices, then one can choose one, say c_1 , of these two heavy cut-vertices so that it is incident to F_1 and one say c_2 that is incident to F_2 . Moreover, this assignment can be done in such a way that if there is a third light cut-vertex c_3 , adjacent to one, say c_1 , of c_1, c_2 , then c_3 is incident to F_1 as well.

Proof. 1. By the third property of Lemma 9, G cannot be contracted to a graph in \mathcal{O}_1 and therefore, from Lemma 6, G must be outerplanar.

2. Let c be a light cut-vertex of a block B in G , with degree 2 (notice that, as c is a cut-vertex, c cannot have degree 1 or 0). That means that c belongs to a path with at least two edges, the hair block B and an edge say e . Observe that $\mathbf{cmp}(G/B) = \mathbf{cmp}(G)$, contradicting to the second property of Lemma 9.

3. Let B be an essential block of G . As it is essential, it has at least one chord, therefore it has at least 2 haploid faces. Assume, that B has at least 3 haploid faces. Choose 3 of them, say F_1, F_2 and F_3 (see Figure 4). Let $S \subseteq E(B)$ be the set of all chords incident to B . Contract in G all edges in $E(G) \setminus S$ not belonging to those faces. Then, for each of the three faces, contract all but two edges not in S that are incident to F_1, F_2 and F_3 and notice that the obtained graph is the graph in \mathcal{O}_2 , a contradiction to the third property of Lemma 9.

4. Let B be a block of G containing more than 3 cut-vertices. Chose four of them, say c_1, c_2, c_3 and c_4 . Let $S \subseteq E(B)$ be the set of all chords incident

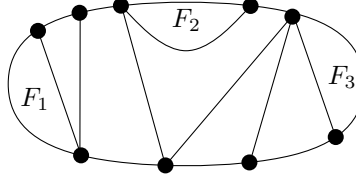


Figure 4: An example for the proof of Lemma 10.3.

to B (see Figure 5). Contract all edges in $E(G) \setminus S$ not having an endpoint in $\{c_1, c_2, c_3, c_4\}$. Then, contract all edges $e \in E(B) \setminus S$ such that $e \not\subseteq \{c_1, c_2, c_3, c_4\}$ and all edges not in $E(B)$, except from one for each of the cut-vertices. Notice that the obtained graph belongs to \mathcal{O}_3 , a contradiction to the third property of Lemma 9.

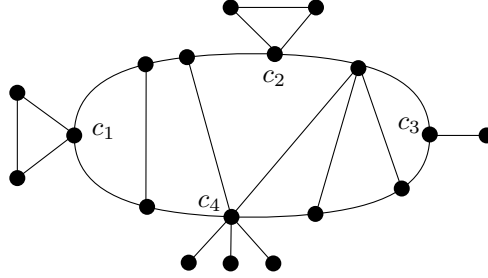


Figure 5: An example for the proof of Lemma 10.4.

5. Let B be a block of G containing a cut-vertex c that is not haploid and let $S \subseteq E(B)$ be the set of all chords incident to B (see Figure 6). Contract all edges in $E(G) \setminus E(B)$ not having c as endpoint and all edges in $E(B) \setminus S$ not having c as endpoint, except from two edges for each of the haploid faces. Then contract all edges not in $E(B)$ with c as endpoint, except for one. Notice that the obtained graph belongs to \mathcal{O}_4 , a contradiction to the third property of Lemma 9.

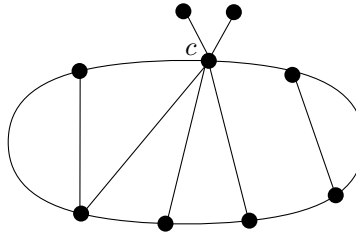


Figure 6: An example for the proof of Lemma 10.5.

6. Let B be a block of G containing three heavy cut-vertices, say c_1, c_2 and c_3 (see Figure 7). We contract all edges in B except from 3 so that B is reduced to a triangle T with vertices c_1, c_2 and c_3 . Then, in the resulting graph H , for each $c_i, i \in \{1, 2, 3\}$, in $\mathcal{C}_H(c_i) \setminus \{T\}$ contains either a non trivial block or at least two hair blocks. In any case, H can be further contracted to one of the graphs in \mathcal{O}_5 a contradiction to the third property of Lemma 9.

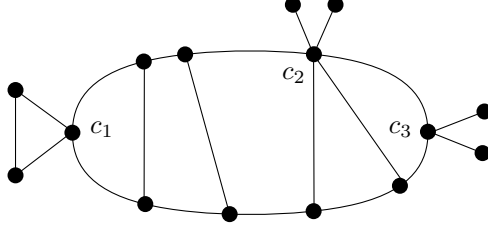


Figure 7: An example for the proof of Lemma 10.6.

7. Let $\{x, y, z\}$ be three cut-vertices of a (not-trivial) block B . If no two of them are connected by an outer edge, then contract all blocks of G , except B , to single edges, then contract all outer edges of B that do not have an endpoint in $\{x, y, z\}$ and continue contracting hair blocks with a vertex of degree ≥ 4 , as long as this is possible. This creates either a graph in \mathcal{O}_6 or a graph that after the contraction of a hair block makes a graph in \mathcal{O}_7 or a graph that after the contraction of two hair blocks is a graph makes a graph in \mathcal{O}_4 and, in any case, we have a contradiction to the third property of Lemma 9. We contract G to a graph H as follows:

- if for some $w \in \{x, y, z\}$ the set $\bar{\mathcal{C}}_G(w, B)$ contains at least two elements, then contract the two of them to a pendant edge (that will have w as an endpoint) and the rest of them to w .
- if for some $w \in \{x, y, z\}$ the set $\bar{\mathcal{C}}_G(w, B)$ contains only one element that is not a hair, then contract it to a triangle (notice that this is always possible because of (2)).

Case 1. $|V(B)| \in \{3, 4\}$. Then because of (6), one, say x of $\{x, y, z\}$ is non-heavy and there is an outer edge connecting x with one, say y , vertex in $\{x, y, z\}$. Then x, y is the required pair of vertices.

Case 2. $|V(B)| > 4$ and there is at most one outer edge e with endpoints from $\{x, y, z\}$ in H . W.l.o.g. we assume that $e = \{x, y\}$. Notice e is a haploid edge, otherwise H can be contracted to the 5th graph in \mathcal{O}_6 . Moreover at least one of x, y is non-heavy, otherwise H can be contracted to one of the graphs in $\mathcal{O}_8 \cup \mathcal{O}_9$ (recall that B may have one or two haploid faces).

Case 3. There are two outer edges with endpoints from $\{x, y, z\}$. W.l.o.g. we assume that these edges are $\{x, y\}$ and $\{y, z\}$. One, say $\{x, y\}$, of $\{x, y\}, \{y, z\}$ is haploid, otherwise H can be contracted to some graph in \mathcal{O}_4 . If $\{x, y\}$ has

8. Let x and y be two heavy cut-vertices vertices of B . From (5) x, y are among the vertices that are incident to the faces F_1 and F_2 . Suppose, in contrary, that for some face, say $F \in \{F_1, F_2\}$, there is no cut vertex in $\{x, y\}$ that is incident to F . Then G can be contracted to one of the graphs in \mathcal{O}_9 . This is enough to prove the first statement except from the case where x and y are both lying in both haploid faces and there is a third light cut-vertex z incident to some, say x , of x, y . In this case, x is assigned the face where z belongs and y is assigned to the other. \square

- $\{X, Y\}$ is a partition of S where X and Y are possibly empty.
- if B has a chord, then all vertices in X and Y are haploid.
- $|X| \leq 1$ and $|Y| \leq 2$.
- If $|Y| = 2$, then its vertices are connected with an edge e and one of them is light and, moreover, in the case where B has a chord then e is haploid.
- If B has a chord, we name the haploid faces of B by F_1 and F_2 such that all vertices in X are incident to F_1 and all vertices of Y are incident to F_2 .

Proof. We examine the non-trivial case where B is a non-trivial block and contains two haploid faces F_1 and F_2 . As B is 2-connected and outer-planar, all vertices of $V(B)$ belong to the unique hamiltonian cycle of B , say C . Our proof is based on the fact that there are exactly two haploid faces and this gives a sense of direction on how the search should be performed. To make this formal, we create an ordering \mathcal{A} of the edges of $E(B)$ using the following procedure.

- 20

10. $e_i = \{u, v\}$
11. $i \leftarrow i + 1$
12. **if** $Q \in E(B)$, **then**
13. $e_i \leftarrow Q$,
14. $i \leftarrow i + 1$
15. **if** $Y \in E(B)$, **then**
16. $e_i \leftarrow Y$

Let $E^{\text{in}}, E^{\text{out}}$ be the edge extensions of $\mathbf{enh}(\mathbf{G}_B)$, and let $\mathbf{prefsec}(\mathcal{A}) = \langle A_0, \dots, A_r \rangle$. It is easy to verify that $\mathcal{E} = \langle E^{\text{in}}, A_0 \cup E^{\text{in}}, \dots, A_r \cup E^{\text{in}} \rangle$ is a monotone and connected $(E^{\text{in}}, E^{\text{out}})$ -expansion of $\mathbf{enh}(\mathbf{G}_B)$, with cost at most 2. \square

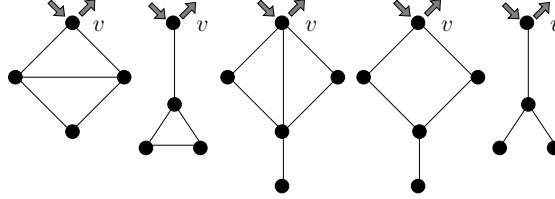


Figure 8: The set \mathcal{A} contains 5 r -graphs, each of the form $(G, \{v\}, \{v\})$.

3.4. Fans

Let G be a graph and v be a vertex in $V(G)$. We denote by $\mathbf{G}^{(v)}$ the rooted graph $(G, \{v\}, \{v\})$ and we refer to it as the *graph G doubly rooted on v* .

We say that a graph G , doubly rooted on some vertex v is a *fan* if none of the graphs in the set \mathcal{A} depicted in Figure 8 is a contraction of the rooted graph $\mathbf{G}^{(v)}$ and G is outerplanar.

Lemma 12. *Let $\mathbf{G}^{(v)} = (G, \{v\}, \{v\})$ be a graph doubly rooted at some vertex v . If $\mathbf{G}^{(v)}$ is a fan, then $\mathbf{cmp}(\mathbf{G}^{(v)}) \leq 2$.*

Proof. We claim first that if $\mathbf{G}^{(v)}$ is a fan, then the graph $G \setminus v$ is a collection of paths where each of them has at least one endpoint that is a neighbour of v . Indeed, if this is not correct, then some of the connected components of $G \setminus v$ would be contractible to either a K_3 or a $K_{1,3}$. In the first case, G is either non-outerplanar or $\mathbf{G}^{(v)}$ can be contracted to the first two rooted graphs of Figure 8. In the second case G is either non-outerplanar or $\mathbf{G}^{(v)}$ the last three graphs of Figure 8. Moreover, if both endpoints of a path in the set of connected components $G \setminus v$ are non adjacent to v in G , then $\mathbf{G}^{(v)}$ can be contracted to the last rooted graph in Figure 8.

Let now P_1, \dots, P_r be the connected components of $G \setminus v$ and, for each $i \in \{1, \dots, r\}$, let $\{v_1^i, \dots, v_{j_i}^i\}$ be the vertices of P_i , ordered as in P_i , such that v_1^i is adjacent to v in G . Let u^{in} and u^{out} be the two vertices added in

$\mathbf{enh}(\mathbf{G}^{(v)})$. If $e^{\text{in}} = \{v, u^{\text{in}}\}$ and $e^{\text{out}} = \{v, u^{\text{out}}\}$ then the edge expansions of $\mathbf{enh}(\mathbf{G}^{(v)})$ is $E^{\text{in}} = \{e^{\text{in}}\}$ and $E^{\text{out}} = \{e^{\text{out}}\}$.

For each $i \in \{1, \dots, r\}$ we define the edge ordering

$$\mathcal{A}_i = \langle \{v, v_1^i\}, \{v_1^i, v_2^i\}, \{v, v_2^i\}, \{v_2^i, v_3^i\} \dots, \{v_{j_i}^i, v\} \rangle,$$

then we delete from \mathcal{A}_i the edges not in $E(G)$. Let \mathcal{A}'_i be the orderings obtained after the edge deletions. We define $\mathcal{A} = \langle e^{\text{in}} \rangle \oplus \mathcal{A}'_1 \oplus \dots \oplus \mathcal{A}'_r$. Notice that $\mathbf{prefsec}(\mathcal{A})$ is a monotone and connected $(E^{\text{in}}, E^{\text{out}})$ -expansion of $\mathbf{enh}(\mathbf{G}^{(v)})$ with cost at most 2. Therefore, $\mathbf{cmp}(\mathbf{G}^{(v)}) \leq 2$ as required. \square

3.5. Spine-degree and central blocks

Given a graph G and a vertex v we denote by $\mathcal{C}_G^{(v)}$ the set of all graphs in $\mathcal{C}_G(v)$, each doubly rooted on v . The *spine-degree* of v is the number of doubly rooted graphs in $\mathcal{C}_G^{(v)}$ that are not fans.

A cut-vertex of a graph G is called *central cut-vertex*, if it has spine-degree greater than 1 and a block of G is called *central block* if it contains at least 2 central cut-vertices.

Lemma 13. *Let $G \in \mathcal{Q}$. The following hold:*

1. *All vertices of G have spine-degree at most 2.*
2. *None of the blocks of G contains more than 2 central cut-vertices.*
3. *G contains at least one central cut-vertex*
4. *There is a total ordering B_1, B_2, \dots, B_r ($r \geq 0$) of the central blocks of G and a total ordering c_1, \dots, c_{r+1} of the central cut-vertices of G such that, for $i \in \{1, \dots, r\}$, the cut-vertices of B_i are c_i and c_{i+1} .*

Proof. 1. Let v be a vertex of G with spine-degree at least 3. That means that there exist at least three subgraphs of G , doubly rooted on v , that can be contracted to some graph in \mathcal{A} , therefore G can be contracted to a graph in \mathcal{O}_{10} , a contradiction.

2. Suppose that B is a block of G containing 3 (or more) central cut-vertices, say c_1, c_2 and c_3 . Construct the graph H by contracting all edges of B to a triangle T with $\{c_1, c_2, c_3\}$ as vertex set. As c_i is a central vertex, there is a rooted graph R_i in $\mathcal{C}_G(c_i)$ that contains some of the graphs in \mathcal{A} as a rooted contraction. Next we apply the same contractions to H for every $c_i, i \in \{1, 2, 3\}$ and then contract to vertices all blocks of H different than T and not contained in some R_i . It is easy to see that the resulting graph is a graph in \mathcal{O}_5 , a contradiction.

3. Assume that G has no central cut-vertices. We distinguish two cases.

Case 1. There is a cut-vertex of G , say v , such that all rooted graphs in $\mathcal{C}_G^{(v)}$ are fans and let $\mathbf{G}_1, \dots, \mathbf{G}_r$ be these rooted graphs. From Lemmata 2 and 12, we conclude that $\mathbf{cmp}(G, \{v\}, \{v\}) = \mathbf{cmp}(\mathbf{glue}(\mathbf{G}_1, \dots, \mathbf{G}_r)) \leq 2$ and from Lemma 1, $\mathbf{cmp}(G) \leq 2$ contradicting to the first condition of Lemma 9.

Case 2. For every cut-vertex v of G , at least one of the rooted graphs in $\mathcal{C}_G^{(v)}$ is not a fan. We denote by H_v the corresponding non-fan rooted graph in $\mathcal{C}_G^{(v)}$ (this is unique due to the fact that v is non-central). Among all cut vertices, let x be one for which the set $V(G) \setminus V(H_x)$ is maximal. Let B be the block of H_x that contains x and let S be the set of the cut-vertices of G that belong to B (including x).

For every $y \in S$ we denote by $\mathcal{W}_y = \{\mathbf{W}_y^1, \dots, \mathbf{W}_y^{r_y}\}$ the set of all rooted graphs in $\mathcal{C}_G^{(y)}$, except from the one, call it \mathbf{R}_y , that contains B . We also define $\mathbf{W}_y = \mathbf{glue}(\mathbf{W}_y^1, \dots, \mathbf{W}_y^{r_y})$. We claim that all $\mathbf{W}_y, y \in S$ are fans. Indeed, if for some y , \mathbf{W}_y is a non-fan, because y is not central, \mathbf{R}_y should be a fan, contradicting the choice of x .

According to the above, the edges of G can be partitioned to those of the rooted graph \mathbf{G}_B and the edges in the rooted graphs $\mathbf{W}_y, y \in S$. Let also $\mathbf{G}_B = (B, X, Y)$ and we assume that, if $Y = \{y_1, y_2\}$, then y_1 is light.

Notice that, according to Lemma 12 $\mathbf{cmp}(\mathbf{W}_y) \leq 2, y \in S$ and according to Lemma 11 $\mathbf{cmp}(\mathbf{G}_B) \leq 2$. We distinguish two cases.

Case 2.1. $Y = \{y_1, y_2\}$ where y_2 is light. Then let $\mathbf{G}_1 = (G[\{y_1, y_2\}], \{y_1, y_2\}, \{y_2\})$ and $\mathbf{G}_2 = (G[\{y_1, y_2\}], \{y_2\}, \{y_1\})$. Clearly $\mathbf{cmp}(\mathbf{G}_1) = 2$ and $\mathbf{cmp}(\mathbf{G}_2) = 1$. Therefore, if $\mathbf{G}' = \mathbf{glue}(\mathbf{G}_B, \mathbf{G}_1, \mathbf{W}_{y_2}, \mathbf{G}_2, \mathbf{W}_{y_1})$, then, from Lemma 2, $\mathbf{cmp}(\mathbf{G}') \leq 2$.

Case 2.2. $Y = \{y_1\}$. Let $\mathbf{G}' = \mathbf{glue}(\mathbf{G}_B, \mathbf{W}_{y_1})$, then, from Lemma 2, $\mathbf{cmp}(\mathbf{G}') \leq 2$.

In both cases, if $X = \{x\}$, then we set $\mathbf{G} = \mathbf{glue}(\mathbf{W}_x, \mathbf{G}')$ while if $X = \emptyset$ we set $\mathbf{G} = \mathbf{G}'$. In any case, we observe that, from Lemma 2, $\mathbf{cmp}(\mathbf{G}) \leq 2$. From Lemma 1, $\mathbf{cmp}(G) \leq 2$ contradicting to the first condition of Lemma 9.

As in both cases we reach a contradiction G must contain at least one central cut-vertex.

4. Let C be the set of all central cut-vertices of G . For each $c \in C$, let \mathcal{X}_c be the subset of $\mathcal{C}_G^{(v)}$ that contains all its members that are not fans. Clearly, \mathcal{X}_c contains exactly two elements. Notice that none of the vertices in $C \setminus \{c\}$ belongs in the double rooted graphs in $\mathcal{C}_G^{(y)} \setminus \mathcal{X}_c$. Indeed, if this is the case for some vertex $y \in C \setminus \{c\}$, then the member of $\mathcal{C}_G^{(y)}$ that avoids c would be a subgraph of some member of $\mathcal{C}_G^{(v)} \setminus \mathcal{X}_c$ and this would imply that some fan would contain as a contraction some double rooted graph that is not a fan. We conclude that for each $c \in C$ there is a partition $p(c) = (A_c, B_c)$ of $C \setminus \{c\}$ such that that all members of A_c are vertices of one of the members of \mathcal{X}_c and all members of B_c are vertices of the other.

We say that a vertex $c \in C$ is *extremal* if $p(c) = \{\emptyset, C \setminus \{c\}\}$

We claim that for any three vertices $\{x, y, z\}$ of C , there is one, say y of them such that x and z belong in different sets of $p(y)$. Indeed, if this is not the case, then one of the following would happen: either there is a vertex $w \in C$ such that x, y, z belong to different elements of $\mathcal{C}_G^{(w)}$, a contradiction to the first statement of this lemma or x, y , and z belong to the same block of G , a contradiction to the second statement of this lemma.

By the above claim, there is a path P containing all central cut-vertices in C and we assume that this path is of minimum length which permits us to assume that its endpoints are extremal vertices of C . Moreover, heavy cut-vertices in $V(P)$ are members of C . Let c_1, \dots, c_{r+1} be the central cut-vertices ordered as they appear in P . As, for every $i \in \{1, \dots, r\}$ there is a block B_i containing the central cut-vertices c_i and c_{i+1} we end up with the two orderings required in the forth statement of the lemma. \square

Let G be a graph in \mathcal{Q} . Suppose also that c_1, \dots, c_{r+1} and B_1, B_2, \dots, B_r are as in Lemma 13.4. We define the *extremal blocks* of G as follows:

- If $r > 0$, then among all blocks that contain c_1 as a cut-vertex let B_0 be the one such that $C_G(c_1, B_0)$, doubly rooted at c_1 , is not a fan, does not contain any edge of the central blocks of G and does not contain c_{r+1} . Symmetrically, among all blocks that contain c_{r+1} as a cut-vertex let B_{r+1} be the one such that $C_G(c_{r+1}, B_{r+1})$ doubly rooted at c_{r+1} is not a fan, does not contain any edge of the central blocks of G and does not contain c_1 .
- If $r = 0$, then let B_0 and B_1 be the two blocks with the property that for $i \in \{0, 1\}$, $C_G(c_1, B_i)$, doubly rooted at c_1 , is not a fan.

We call B_0 and B_{r+1} *left* and *right* extremal block of G respectively. We also call the blocks of G that are either central or extremal *spine* blocks of G . Let $A(G)$ be the set of cut-vertices of the graphs $B_0, B_1, B_2, \dots, B_r, B_{r+1}$. We partition $A(G)$ into three sets A_1, A_2 and A_3 as follows:

- $A_1 = \{c_1, \dots, c_{r+1}\}$ (i.e. all central vertices).
- A_2 contains all vertices of $A(G)$ that belong to central blocks and are not central vertices.
- A_3 contains all vertices of $A(G)$ that belong to extremal blocks and are not central cut-vertices.

Moreover, we further partition A_3 to two sets $A_3^{(0)} = A_3 \cap V(B_0)$ and $A_3^{(r+1)} = A_3 \cap V(B_{r+1})$.

Let $G \in \mathcal{Q}$ and $v \in A(G)$. We denote by $\mathcal{R}_G^{(v)}$ the set of the doubly rooted graphs in $\mathcal{C}_G^{(v)}$ that do not contain any of the edges of the spine blocks of G .

Lemma 14. *Let $G \in \mathcal{Q}$ and $v \in A(G)$. The following hold:*

- Each doubly rooted graph in $\mathcal{R}_G^{(v)}$ is a fan.*
- All vertices in $v \in A_2$ are light, i.e., for each $v \in A_2$ $\mathcal{R}_G^{(v)}$ contains exactly one graph that is a hair block of G .*

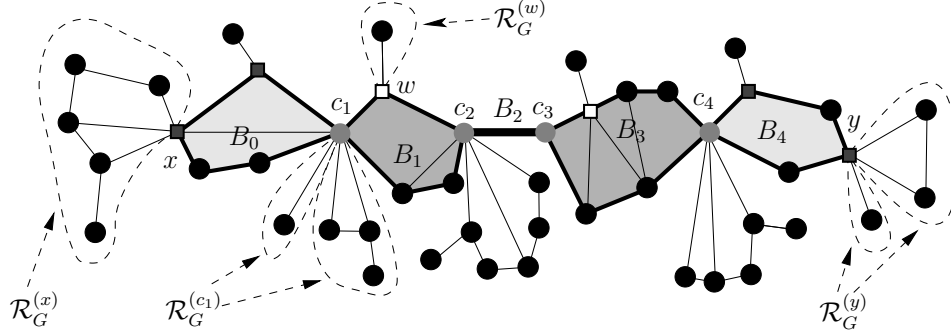


Figure 9: A graph G and the blocks B_0, B_1, \dots, B_3 , and B_4 . The cut-vertices in $A_1 = \{c_1, \dots, c_4\}$ are the grey circular vertices, the vertices in A_2 are the white square vertices and the vertices in A_3 are the dark square vertices.

Proof. a) Let $v \in A(G)$. We distinguish two cases:

Case 1: $v \in A_1$. If there exist a double rooted graph in $\mathcal{R}_G^{(v)}$ that is not a fan, v will have spine-degree greater than 3, a contradiction to the first property of Lemma 13.

Case 2: $v \in A_2 \cup A_3$. If there exist a double rooted graph in $\mathcal{R}_G^{(v)}$ that is not a fan, v will have spine-degree greater than 2, therefore v must be central, a contradiction.

b) Let $v \in A_2$, and suppose that $\mathcal{R}_G^{(v)}$ can be contracted to two edges with v as their unique common endpoint, or to a triangle. As v belongs to a central block, G can be contracted to a graph in \mathcal{O}_5 , a contradiction to the third property of Lemma 9. \square

3.6. Directional obstructions

Let $G \in \mathcal{Q}$ and let $B_0, B_1, \dots, B_r, B_{r+1}$ be the spine blocks of G . Notice first that from Lemma 10.6 for every $i \in \{1, \dots, r\}$, $|A_2 \cap V(B_i)| \leq 1$. Also, from Lemma 10.6, if $A_2 \cap V(B_i) = \{v\}$, then v is a light cut-vertex. For $i \in \{1, \dots, r\}$, we define the rooted graphs \mathbf{B}_i^* as follows: if $A_2 \cap V(B_i) = \{v\}$, then \mathbf{B}_i^* is the union of B_i and the underlying graph of the unique rooted graph in $\mathcal{R}_G^{(v)}$ (this rooted graph is unique and its underlying graph is a hair block of G from the second statement of Lemma 14). If $A_2 \cap V(B_i) = \emptyset$, then \mathbf{B}_i^* is B_i . We finally define the rooted graph $\mathbf{B}_i^* = (B_i^*, \{c_i\}, \{c_{i+1}\})$ for $i \in \{1, \dots, r\}$.

We also define B_0^* as follows: consider the unique graph B_0 in $\mathcal{C}_G^{(c_1)}$ that, when doubly rooted on c_1 , is not a fan, does not contain any edge of the central blocks of G and does not contain c_{r+1} . Then $\mathbf{B}_0^* = (B_0, \emptyset, \{c_1\})$. Analogously, we define B_{r+1}^* by considering the graph B_{r+1} of the unique rooted

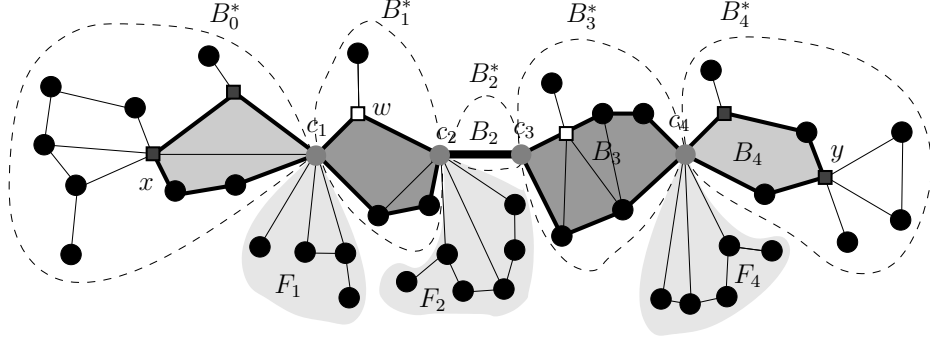


Figure 10: A graph G , the extended extremal blocks B_0^* and B_4^* the extended central blocks B_1^* , B_2^* , and B_3^* and the rooted graphs F_1, F_2 and F_4 (F_3 is the graph consisting only of the vertex c_3 , doubly rooted on c_3).

graph in $\mathcal{R}_G^{(c_{r+1})}$ that, when doubly rooted on c_{r+1} , is not a fan, does not contain any edge of the central blocks of G and does not contain c_1 . Then $\mathbf{B}_{r+1}^* = (B_{r+1}, \{c_{r+1}\}, \emptyset)$. Finally, we define for each $i \in \{1, \dots, r+1\}$ the graph F_i that is the union of all the graphs of the rooted graphs in $\mathcal{R}_G^{(c_i)}$ that are fans (when performing the union, the vertex c_i stays the same), and in the case where $\mathcal{R}_G^{(c_i)}$ is empty, then F_i is the trivial graph $(\{c_i\}, \emptyset)$. We set $\mathbf{F}_i = (F_i, \{c_i\}, \{c_i\})$ $i \in \{1, \dots, r+1\}$ and we call the rooted graphs $\mathbf{F}_1, \dots, \mathbf{F}_{r+1}$ *extended fans* of G . We call $\mathbf{B}_0^*, \mathbf{B}_1^*, \dots, \mathbf{B}_r^*, \mathbf{B}_{r+1}^*$ the *extended blocks* of the graph $G \in \mathcal{Q}$ and we naturally distinguish them in *central* and *extremal* (left or right), depending of type of the blocks that contain them. Notice that

$$\{E(B_0^*), E(F_1), E(B_1^*), E(F_2), \dots, E(F_r), E(B_r^*), E(F_{r+1}), E(B_{r+1}^*)\}$$

is a partition of the edges of G .

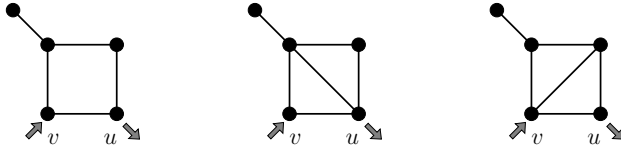


Figure 11: The set of rooted graphs \mathcal{L} containing three rooted graphs each of the form $(G, \{v\}, \{u\})$.

Lemma 15. *Let $G \in \mathcal{Q}$ and let $\mathbf{B}_1^*, \dots, \mathbf{B}_r^*$ be the central blocks of G . None of the rooted graphs in the set \mathcal{L} of Figure 11 is a contraction of \mathbf{B}_i^* if and only if $\text{cmp}(\mathbf{B}_i^*) \leq 2$.*

Proof. Clearly, for every graph $H \in \mathcal{L}$, $\mathbf{cmp}(H, \{v\}, \{u\}) = 3$, therefore if \mathbf{B}_i^* can be contracted to a graph in \mathcal{L} , according to Lemma 4, $\mathbf{cmp}(\mathbf{B}_i^*) \geq 3$.

Let now \mathbf{B}_i^* be an central extended block of G that cannot be contracted to a graph in \mathcal{L} . If B_i does not contain some light cut-vertex, we define $\mathbf{G}_{B_i} = (B_i, \{c_i\}, \{c_{i+1}\})$. Notice that $\mathbf{B}_i^* = \mathbf{G}_{B_i}$ and, as from Lemma 11 $\mathbf{cmp}(\mathbf{G}_{B_i}) \leq 2$, we are done.

In the remaining case, where B_i contains a light cut-vertex, say c , observe that c cannot be adjacent via an outer edge to c_i , or else B_i^* could be contracted to a graph in \mathcal{L} . Therefore, according to Lemma 10.7, c is connected via an haploid edge with c_{i+1} . Notice that $\mathbf{G}_{B_i} = (B_i, \{c_i\}, \{c_{i+1}, c\})$. According to Lemma 11, $\mathbf{cmp}(\mathbf{G}_{B_i}) \leq 2$ and, according to Lemma 14, $\mathcal{R}^{(c)}$ contains only a hair block, say $(H, \{c\}, \{c\})$. Clearly $\mathbf{cmp}(H, \{c\}, \{c\}) = 2$. Let $\mathbf{G}_1 = (G[\{c, c_{i+1}\}], \{c, c_{i+1}\}, \{c\})$, $\mathbf{G}_2 = (G[\{c, c_{i+1}\}], \{c\}, \{c_{i+1}\})$, and $\mathbf{G} = \mathbf{glue}(\mathbf{G}_{B_i}, \mathbf{G}_1, (H, \{c\}, \{c\}), \mathbf{G}_2)$. From Lemma 2, $\mathbf{cmp}(\mathbf{G}) \leq 2$ and the lemma follows as $\mathbf{G} = \mathbf{B}_i^*$. \square

Lemma 16. *If \mathbf{B}_i^* is one of the central extended blocks of a graph $G \in \mathcal{Q}$, then either $\mathbf{cmp}(\mathbf{B}_i^*) \leq 2$ or $\mathbf{cmp}(\mathbf{rev}(\mathbf{B}_i^*)) \leq 2$.*

Proof. Proceeding towards a contradiction, from Lemma 15, both \mathbf{B}_i^* and $\mathbf{rev}(\mathbf{B}_i^*)$ must contain one of the rooted graphs in Figure 11 as a contraction. It is easy to verify that, in this case, either B_i^* contains at least four cut-vertices, which contradicts to Lemma 10.9 or G can be contracted to either a graph in \mathcal{O}_8 (if the two roots are adjacent) or a graph in \mathcal{O}_6 (if the two roots are not adjacent), a contradiction to Lemma 9.3. \square

Let $G \in \mathcal{Q}$ and let \mathbf{B}_i^* be one of the central extended blocks of G .

- If $\mathbf{rev}(\mathbf{B}_i^*)$ can be contracted to a graph in \mathcal{L} , then we assign to \mathbf{B}_i^* the label \leftarrow .
- If \mathbf{B}_i^* can be contracted to a graph in \mathcal{L} , then we assign to \mathbf{B}_i^* the label \rightarrow .
- If both \mathbf{B}_i^* and $\mathbf{rev}(\mathbf{B}_i^*)$ can be contracted to a graph in \mathcal{L} , then we assign to \mathbf{B}_i^* the label \leftrightarrow .

Lemma 17. *Let $G \in \mathcal{Q}$ and let \mathbf{B}_0^* be the extended left extremal block of G . None of the rooted graphs in the set \mathcal{B} in Figure 12 is a contraction of \mathbf{B}_0^* if and only if $\mathbf{cmp}(\mathbf{B}_0^*) \leq 2$.*

Proof. Clearly, for every graph $H \in \mathcal{B}$, $\mathbf{cmp}(H, \emptyset, \{u\}) = 3$. Therefore, if \mathbf{B}_0^* can be contracted to a graph in \mathcal{B} , according to Lemma 4, $\mathbf{cmp}(\mathbf{B}_0^*) \geq 3$.

Suppose now that \mathbf{B}_0^* cannot be contracted to a graph in \mathcal{B} . We distinguish three cases according to the number of cut-vertices in B_0 (recall that, from Lemma 10.4, B_0 can have at most 3 cut-vertices).

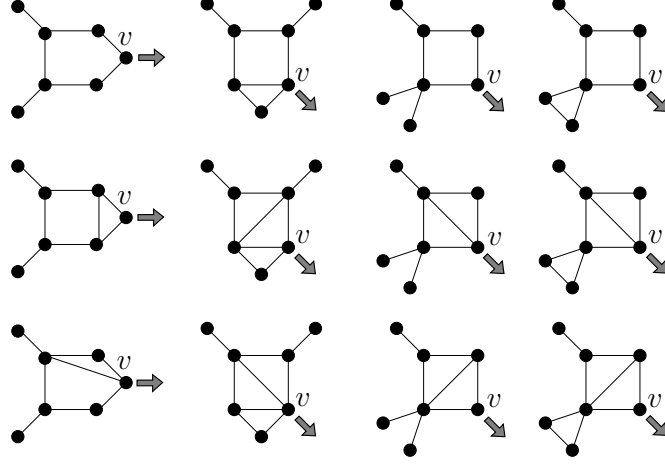


Figure 12: The set of rooted graphs \mathcal{B} containing 12 rooted graphs each of the form $(G, \emptyset, \{v\})$.

Case 1: B_0 contains only one cut-vertex, which is c_1 . Then, $\mathbf{B}_0^* = \mathbf{G}_{B_0}$ and the result follows because of Lemma 11.

Case 2: B_0 contains two cut-vertices, c_1 and c . If B_0 has not a chord or it has a chord and c and c_1 are incident to two different haploid faces of B_0 then we can assume that $\mathbf{G}_{B_0} = (B_0, \{c\}, \{c_1\})$ and from Lemma 11, $\mathbf{cmp}(\mathbf{G}_{B_0}) \leq 2$. According to Lemma 14.a, $\mathcal{R}^{(c)}$ is a fan, say $(F, \{c\}, \{c\})$ and from Lemma 12 $\mathbf{cmp}(F, \{c\}, \{c\}) \leq 2$. Let $\mathbf{G} = \mathbf{glue}((F, \{c\}, \{c\}), \mathbf{G}_{B_0})$. From Lemma 2, $\mathbf{cmp}(\mathbf{G}) \leq 2$. Combining the fact that $\mathbf{G} = (B_0^*, \{c\}, \{c_1\})$ with Lemma 1, $\mathbf{cmp}(\mathbf{B}_0^*) \leq 2$. In the remaining case c and c_1 are adjacent and c is light. Then $\mathbf{G}_{B_0} = (B_0, \emptyset, \{c, c_1\})$ and, from Lemma 11, $\mathbf{cmp}(\mathbf{G}_{B_0}) \leq 2$. According to Lemma 14.b, $\mathcal{R}^{(c)}$ is a hair, say $(H, \{c\}, \{c\})$. Let $\mathbf{G}_1 = (G[\{c, c_1\}], \{c, c_1\}, \{c\})$, $\mathbf{G}_2 = (G[\{c, c_1\}], \{c\}, \{c_1\})$ and $\mathbf{G} = \mathbf{glue}(\mathbf{G}_{B_0}, \mathbf{G}_1, (H, \{c\}, \{c\}), \mathbf{G}_2)$. Notice that $\mathbf{G} = (B_0^*, \emptyset, \{c_1\}) = \mathbf{B}_0^*$. From Lemma 2, we obtain that $\mathbf{cmp}(\mathbf{G}) \leq 2$, therefore $\mathbf{cmp}(\mathbf{B}_0^*) \leq 2$.

Case 3: B_0 contains three cut-vertices, c_1 , c and x . We first examine the case where there is a partition $\{X, Y\}$ of $\{c_1, c, x\}$ such that $|Y| = 2$, $c_1 \in Y$, the cut-vertex in $Y \setminus \{c_1\}$ is light, and Y is an edge of B_0 that, in case B_0 is a chord, is haploid. In this case we claim that $\mathbf{cmp}(\mathbf{B}_0^*) \leq 2$. Indeed, we may assume that c be the light cut-vertex of $Y \setminus \{c_1\}$, thus $\mathbf{G}_{B_0} = (B_0, \{x\}, \{c, c_1\})$. According to Lemma 11, $\mathbf{cmp}(\mathbf{G}_{B_0}) \leq 2$. From Lemma 14.a, $\mathcal{R}^{(x)}$ is a fan, say $(F, \{x\}, \{x\})$ and, from Lemma 14.b, $\mathcal{R}^{(c)}$ contains only a hair block, say $(H, \{c\}, \{c\})$. Let $\mathbf{G}_1 = (G[\{c, c_1\}], \{c, c_1\}, \{c\})$, $\mathbf{G}_2 = (G[\{c, c_1\}], \{c\}, \{c_1\})$ and $\mathbf{G} = \mathbf{glue}((F, \{x\}, \{x\}), \mathbf{G}_{B_0}, \mathbf{G}_1, (H, \{c\}, \{c\}), \mathbf{G}_2)$. Notice that $\mathbf{G} = (B_0^*, \{x\}, \{c_1\})$ and. From Lemma 2 $\mathbf{cmp}(\mathbf{G}) \leq 2$. Now, Lemma 1 implies that

$\mathbf{cmp}(\mathbf{B}_0^*) \leq 2$ and the claim holds.

In the remaining cases, the following may happen:

1. None of x and c is adjacent to c_1 via an edge that, in case B_0 has a chord, is haploid. In this case \mathbf{B}_0^* can be contracted to the rooted graphs of the first column in Figure 12.

2. Both c_1 and x are light and only one of them, say x , is adjacent to c_1 . In this case B_0 has a chord and either the edge $\{x, c_1\}$ is not haploid or $\{x, c_1\}$ is haploid and belongs in the same haploid face with c . In the first case, \mathbf{B}_0^* can be contracted to the second rooted graph of the second column in Figure 12 and in the second case \mathbf{B}_0^* can be contracted to the first and the third rooted graph of the second column in Figure 12.

3. c_1 has only one, say x , heavy neighbour in $\{c, x\}$ such that, in case B_0 has a chord, the edge $\{c_1, x\}$ is haploid. In this case \mathbf{B}_0^* can be contracted to the rooted graphs of the third and the fourth column in Figure 12. \square

Lemma 18. *Let $G \in \mathcal{Q}$ and let \mathbf{B}_{r+1}^* be the extended right extremal block of G . None of the rooted graphs in the set \mathcal{C} in Figure 13 is a contraction of \mathbf{B}_{r+1}^* if and only if $\mathbf{cmp}(\mathbf{B}_{r+1}^*) \leq 2$.*

Proof. Clearly, for every graph $H \in \mathcal{C}$, $\mathbf{cmp}(H, \{u\}, \emptyset) = 3$, therefore if \mathbf{B}_{r+1}^* can be contracted to a graph in \mathcal{C} , according to Lemma 4, $\mathbf{cmp}(\mathbf{B}_{r+1}^*) \geq 3$.

Suppose now that \mathbf{B}_{r+1}^* cannot be contracted to a graph in \mathcal{C} . We distinguish three cases according to the number of cut-vertices in B_{r+1} (recall that, from Lemma 10.4, B_{r+1} can have at most 3 cut-vertices).

Case 1: B_{r+1} contains only one cut-vertex, which of course is c_1 . Notice that $\mathbf{B}_{r+1}^* = \mathbf{G}_{B_{r+1}}$ and the result follows because of Lemma 11.

Case 2: If B_{r+1} has not a chord or it has a chord and c and c_1 are incident to two different haploid faces of B_{r+1} then we can assume that $\mathbf{G}_{B_{r+1}} = (B_{r+1}, \{c_{r+1}\}, \{c\})$ and from Lemma 11, $\mathbf{cmp}(\mathbf{G}_{B_{r+1}}) \leq 2$. In any other case, c_{r+1} and c is on the boundary in the same haploid face of B_{r+1} and none of them belongs in the boundary of the other. Then \mathbf{B}_{r+1}^* can be contracted to some of the rooted graphs in the second column of Figure 13.

Case 3: B_{r+1} contains three cut-vertices, c_{r+1} , c and x . If c and x are not adjacent, then \mathbf{B}_{r+1}^* can be contracted to some of the rooted graphs in the first column of Figure 13. Otherwise, one, say c , of them will be light and the edge $\{x, c\}$ should be haploid. Then we can assume that $\mathbf{G}_{B_{r+1}} = (B_{r+1}, \{c_{r+1}\}, \{x, c\})$ and according to Lemma 11, $\mathbf{cmp}(\mathbf{G}_{B_{r+1}}) \leq 2$. From Lemma 14, $\mathcal{R}^{(x)}$ is a fan, say $(F, \{x\}, \{x\})$ and $\mathcal{R}^{(c)}$ contains only a hair block, say $(H, \{c\}, \{c\})$. Let $\mathbf{G}_1 = (G[\{c, x\}], \{c, x\}, \{c\})$, $\mathbf{G}_2 = (G[\{c, x\}], \{c\}, \{x\})$ and $\mathbf{G} = \mathbf{glue}(\mathbf{G}_{B_{r+1}}, \mathbf{G}_1, (H, \{c\}, \{c\}), \mathbf{G}_2, (F, \{x\}, \{x\}))$. Notice that $\mathbf{G} = (B_{r+1}, \{c_{r+1}\}, \{x\})$ and, because of Lemma 2, $\mathbf{cmp}(\mathbf{G}) \leq 2$. Applying Lemma 1, we conclude that $\mathbf{cmp}(\mathbf{B}_{r+1}^*) \leq 2$. \square

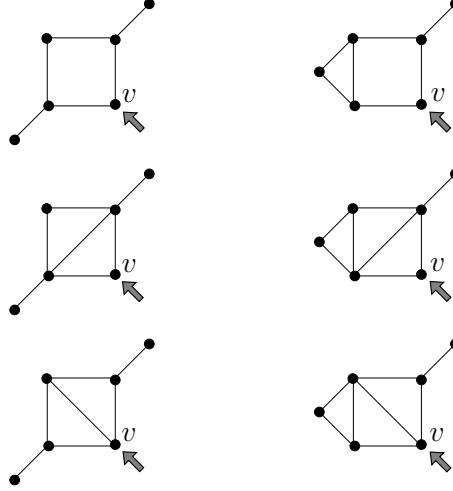


Figure 13: The set of rooted graphs \mathcal{C} containing six rooted graphs each of the form $(G, \{v\}, \emptyset)$.

Lemma 19. *Let $G \in \mathcal{Q}$ and let \mathbf{B}_0^* and \mathbf{B}_{r+1}^* be the two extremal extended blocks of G . It is never the case that \mathbf{B}_0^* contains some graph in \mathcal{B} and $\text{rev}(\mathbf{B}_0^*)$ contains some rooted graph in \mathcal{C} . Also it is never the case that \mathbf{B}_{r+1}^* contains some graph in \mathcal{C} and $\text{rev}(\mathbf{B}_{r+1}^*)$ contains some rooted graph in \mathcal{B} .*

Proof. Let \mathbf{G} be a rooted graph in $K = \{\text{rev}(\mathbf{B}_0^*), \mathbf{B}_{r+1}^*\}$. We distinguish the following cases, that apply for both rooted graphs in K :

Case 1: \mathbf{G} can be contracted to a graph in the first column of Figure 13 and $\text{rev}(\mathbf{G})$ to a graph in the first column of Figure 12. Notice that every cut-vertex of \mathbf{G} cannot be connected with an outer edge and therefore \mathbf{G} can be contracted to graph in \mathcal{O}_6 .

Case 2: \mathbf{G} can be contracted to a graph in the first column of Figure 13 and $\text{rev}(\mathbf{G})$ to a graph in the second column of Figure 12. Notice that the two cut-vertices of \mathbf{G} , that are other than the central cut-vertex, cannot be connected with an outer edge and therefore \mathbf{G} can be contracted to graph in \mathcal{O}_7 .

Case 3: \mathbf{G} can be contracted to a graph in the first column of Figure 13 and $\text{rev}(\mathbf{G})$ to a graph in the third or fourth column of Figure 12. Notice that the two cut-vertices of \mathbf{G} , that are other than the central cut-vertex, cannot be connected with an outer edge and therefore \mathbf{G} can be contracted to graph in \mathcal{O}_8 .

Case 4: \mathbf{G} can be contracted to a graph in the second column of Figure 13 and $\text{rev}(\mathbf{G})$ to a graph in the first column of Figure 12. Notice that the two cut-vertices of \mathbf{G} , that are other than the central cut-vertex, cannot be connected

with an outer edge and therefore \mathbf{G} can be contracted to graph in \mathcal{O}_7 .

Case 5: \mathbf{G} can be contracted to a graph in the second column of Figure 13 and $\mathbf{rev}(\mathbf{G})$ to a graph in the second column of Figure 12. Notice that there must be an haploid face containing only the central cut-vertex, therefore \mathbf{G} can be contracted either to the graph in \mathcal{O}_2 or to a graph in \mathcal{O}_4 (depending whether \mathbf{G} can be contracted to the last graph in the second column of Figure 13 or not).

Case 6: \mathbf{G} can be contracted to a graph in the second column of Figure 13 and $\mathbf{rev}(\mathbf{G})$ to a graph in the third or fourth column of Figure 12. Notice that the light cut-vertex of \mathbf{G} can not be connected via an haploid edge with the central cut-vertex, therefore \mathbf{G} can be contracted to a graph in \mathcal{O}_7 either to a graph in \mathcal{O}_9 (depending whether the central cut-vertex is connected via haploid edge with a heavy cut-vertex or not). \square

- If \mathbf{B}_0^* contains some graph in \mathcal{B} then we assign to \mathbf{B}_0^* the label \leftarrow .
- If $\mathbf{rev}(\mathbf{B}_0^*)$ contains some graph in \mathcal{C} then we assign to \mathbf{B}_0^* the label \rightarrow .
- If \mathbf{B}_{r+1}^* contains some graph in \mathcal{C} then we assign to \mathbf{B}_{r+1}^* the label \leftarrow .
- If $\mathbf{rev}(\mathbf{B}_{r+1}^*)$ contains some graph in \mathcal{B} then we assign to \mathbf{B}_{r+1}^* the label \rightarrow .
- If neither \mathbf{B}_0^* contains some graph in \mathcal{B} nor $\mathbf{rev}(\mathbf{B}_0^*)$ contains some graph in \mathcal{C} then we assign to \mathbf{B}_0^* the label \leftrightarrow .
- If neither \mathbf{B}_{r+1}^* contains some graph in \mathcal{C} nor $\mathbf{rev}(\mathbf{B}_{r+1}^*)$ contains some graph in \mathcal{B} then we assign to \mathbf{B}_{r+1}^* the label \leftrightarrow .

Lemma 20. *Let $G \in \mathcal{Q}$ and let $\mathbf{B}_0^*, \mathbf{B}_1^*, \dots, \mathbf{B}_r^*, \mathbf{B}_{r+1}^*$ be the extended blocks of G . It is not possible that one of these extended blocks is labeled with \leftarrow and another with \rightarrow .*

Proof. We distinguish two cases according to the labelling of \mathbf{B}_0^* :

Case 1: Suppose that \mathbf{B}_0^* is labeled \leftarrow and that \mathbf{B}_i^* , for some $i \in \{1, \dots, r+1\}$, is labeled \rightarrow . According to their respective labels, $(\mathbf{B}_0^*, \emptyset, \{c_1\})$ can be contracted to a graph in \mathcal{B} and, if $i \leq r$, $\mathbf{rev}(\mathbf{B}_i^*) = (B_i^*, \{c_{i+1}\}, \{c_i\})$ can be contracted to a graph in \mathcal{L} , otherwise $\mathbf{rev}(\mathbf{B}_{r+1}^*) = (B_{r+1}^*, \emptyset, \{c_{r+1}\})$ can be contracted to a rooted graph in \mathcal{B} . Notice that if $i \leq r$, $G[V(G) \setminus (V(B_1^*) \cup \dots \cup V(B_{i-1}^*))], \emptyset, \{c_i\}$ can be contracted to a graph in the third and forth columns of \mathcal{B} . By further contracting all edges of $E(B_1^*) \cup \dots \cup E(B_{i-1}^*)$ we obtain a graph in \mathcal{O}_{11} , a contradiction.

Case 2: Suppose now that \mathbf{B}_0^* is labeled \rightarrow and that \mathbf{B}_i^* , for some $i \in \{1, \dots, r+1\}$, is labeled \leftarrow . According to their respective labels, $\mathbf{rev}(\mathbf{B}_0^*) = (B_0^*, \{c_1\}, \emptyset)$ can be contracted to a graph in \mathcal{C} and, if $i \leq r$, $(B_i^*, \{c_i\}, \{c_{i+1}\})$ can be contracted to a graph in \mathcal{L} , otherwise $(B_{r+1}^*, \{c_{r+1}\}, \emptyset)$ can be contracted to a graph

in \mathcal{C} . Notice that if $i \leq r$, $G[V(G) \setminus (V(B_1^*) \cup \dots \cup V(B_{i-1}^*))], \{c_i\}, \emptyset$ can be contracted to a graph in the first column of \mathcal{C} . By further contracting all edges of $E(B_1^*) \cup \dots \cup E(B_{i-1}^*)$ we obtain a graph in \mathcal{O}_{12} , a contradiction. \square

3.7. Putting things together

Lemma 21. $\mathcal{Q} = \emptyset$.

Proof. Suppose in contrary that \mathcal{Q} contains some graph G . Let $\mathbf{B}_0^*, \mathbf{B}_1^*, \dots, \mathbf{B}_r^*, \mathbf{B}_{r+1}^*$ and $\mathbf{F}_1, \dots, \mathbf{F}_{r+1}$ be the extended blocks and fans of G , respectively. From Lemma 20, we can assume that the extended blocks of G are all labeled either \rightarrow or \leftrightarrow (if this is not the case, just reverse the ordering of the blocks). By the labelling of \mathbf{B}_0^* , none of the rooted graphs in the set \mathcal{B} is a contraction of \mathbf{B}_0^* therefore, from Lemma 17, $\mathbf{cmp}(\mathbf{B}_0^*) \leq 2$. Also as none of the rooted graphs \mathbf{B}_i^* , $i = 1, \dots, r$, can be contracted to a graph in \mathcal{L} , from Lemma 15, it follows that $\mathbf{cmp}(\mathbf{B}_i^*) \leq 2$. We distinguish two cases according to the labelling of \mathbf{B}_{r+1}^* . If the labelling is \leftrightarrow , then \mathbf{B}_{r+1}^* cannot be contracted to a graph in \mathcal{C} . If the labelling is \rightarrow , then $\mathbf{rev}(\mathbf{B}_{r+1}^*)$ can be contracted to a graph in \mathcal{B} and, according to Lemma 20, \mathbf{B}_{r+1}^* cannot be contracted to a graph in \mathcal{C} . Thus, in both cases, from Lemma 18, $\mathbf{cmp}(\mathbf{B}_{r+1}^*) \leq 2$. Notice that $(G, \emptyset, \emptyset) = \mathbf{glue}(\mathbf{B}_0^*, \mathbf{F}_1, \mathbf{B}_1^*, \dots, \mathbf{F}_r, \mathbf{B}_r^*, \mathbf{F}_{r+1}, \mathbf{B}_{r+1}^*)$ and, from Lemma 2, $\mathbf{cmp}(G, \emptyset, \emptyset) \leq 2$. This implies that $\mathbf{cmp}(G) \leq 2$, a contradiction to the first property of Lemma 9. \square

Corollary 1. $\mathbf{obs}_{\preceq}(\mathcal{G}[\mathbf{cmms}, 2]) = \mathbf{obs}_{\preceq}(\mathcal{G}[\mathbf{cms}, 2])$.

Proof. It is easy to check that for every $H \in \mathbf{obs}_{\preceq}(\mathcal{G}[\mathbf{cmms}, 2])$:

1. $\mathbf{cms}(H) \geq 3$,
2. for every proper contraction H' of H it holds that $\mathbf{cms}(H') \leq 2$,

therefore $\mathbf{obs}_{\preceq}(\mathcal{G}[\mathbf{cmms}, 2]) \subseteq \mathbf{obs}_{\preceq}(\mathcal{G}[\mathbf{cms}, 2])$.

If there exist a graph $H \in \mathbf{obs}_{\preceq}(\mathcal{G}[\mathbf{cms}, 2]) \setminus \mathbf{obs}_{\preceq}(\mathcal{G}[\mathbf{cmms}, 2])$, then $\mathbf{cms}(H) \geq 3$. Notice that the connected search number of a graph is always bounded from the monotone and connected search number, as a complete monotone and connected search strategy is obviously a complete connected search strategy, therefore $\mathbf{cmms}(H) \geq 3$, which means that there exist a graph $H' \in \mathbf{obs}_{\preceq}(\mathcal{G}[\mathbf{cmms}, 2])$ such that $H' \preceq H$. Furthermore, since H' is a proper contraction of H , according to Lemma 5, $\mathbf{cms}(H) \geq \mathbf{cms}(H')$. As we have already stressed that $\mathbf{cms}(H') \geq 3$ we reach a contradiction to the minimality (with respect of \preceq) of H . \square

4. General Obstructions for cmms

As we mentioned before, for $k > 2$, we have no guarantee that the set $\mathbf{obs}_{\preceq}(\mathcal{G}[\mathbf{cmms}, k])$ is a finite set. In this section we prove that when this set is finite its size should be double exponential in k . Therefore, it seems hard to extend our results for $k \geq 3$ as, even if we somehow manage to prove that the

obstruction set for a specific k is finite, then this set would contain more than $2^{2^{\Omega(k)}}$ graphs.

We will describe a procedure that generates, for each k , a set of at least $\frac{4}{3}(\frac{5}{2})^{3 \cdot 2^{k-2}}$ non-isomorphic graphs that have connected and monotone search number $k+1$ and are contraction-minimal with respect to this property. Hence, these $\frac{4}{3}(\frac{5}{2})^{3 \cdot 2^{k-2}}$ graphs will belong to $\mathbf{obs}_{\preceq}(\mathcal{G}[\mathbf{cmms}, k])$.

We define for every $k \geq 1$ a set of rooted graphs, namely the set of *obstruction-branches* denoted $\mathbf{Br}(k)$, as follows:

For $k = 1$: The set $\mathbf{Br}(1)$ consists of the five graphs of Figure 8 rooted at v .

For $k = l > 1$: The set $\mathbf{Br}(l)$ is constructed by choosing two *branches* of the set $\mathbf{Br}(l-1)$ and identify the two roots to a single vertex, say v . Then we add a new edge with v as an endpoint, say $\{u, v\}$, and we root this branch to u . We will refer to this edge as the *trunk* of the branch.

Let $f(k)$ be the number of branches of $\mathbf{Br}(k)$. Notice that $f(1) = 5$ and $f(k)$ is equal to the number of ways we can pick two branches of $\mathbf{Br}(k-1)$, with repetition. Therefore:

$$\begin{aligned} f(k) &= \binom{f(k-1) + 2 - 1}{2} = \binom{f(k-1) + 1}{2} \\ &= \frac{f(k-1)^2 + f(k-1)}{2} \geq \frac{f(k-1)^2}{2} \\ &\geq \frac{\left(\frac{f(k-2)^2}{2}\right)^2}{2} = \frac{f(k-2)^{2^2}}{2^{2+1}} \geq \frac{f(k-3)^{2^3}}{2^{2^2+2+1}} \\ &\geq \dots \geq \frac{f(1)^{2^{k-1}}}{2^{2^{k-2}+\dots+2+1}} = \frac{5^{2^{k-1}}}{2^{2^k-1}} = 2\left(\frac{5}{2}\right)^{2^{k-1}} \end{aligned}$$

Let $\mathcal{O}_{\mathbf{Br}}(k)$ be the set containing the graphs obtained by choosing three rooted branches of $\mathbf{Br}(k)$, with repetitions, and identify the three roots. Notice that any two such selections produce two non-isomorphic graphs. We are going to prove that $\mathcal{O}_{\mathbf{Br}}(k) \subseteq \mathbf{obs}_{\preceq}(\mathcal{G}[\mathbf{cmms}, k+1])$. Notice that:

$$\begin{aligned} |\mathcal{O}_{\mathbf{Br}}(k)| &= \binom{f(k) + 3 - 1}{3} = \binom{f(k) + 2}{3} \\ &= \frac{(f(k) + 2)(f(k) + 1)f(k)}{6} \\ &= \frac{f(k)^3 + 3f(k)^2 + 2f(k)}{6} \geq \frac{f(k)^3}{6} \\ &\geq \frac{4}{3}\left(\frac{5}{2}\right)^{3 \cdot 2^{k-1}} \end{aligned}$$

Hence the cardinality of $\mathbf{obs}_{\preceq}(\mathcal{G}[\mathbf{cmms}, k])$ is at least $\frac{4}{3}\left(\frac{5}{2}\right)^{3 \cdot 2^{k-2}}$. In order to prove this we need the following Lemmata.

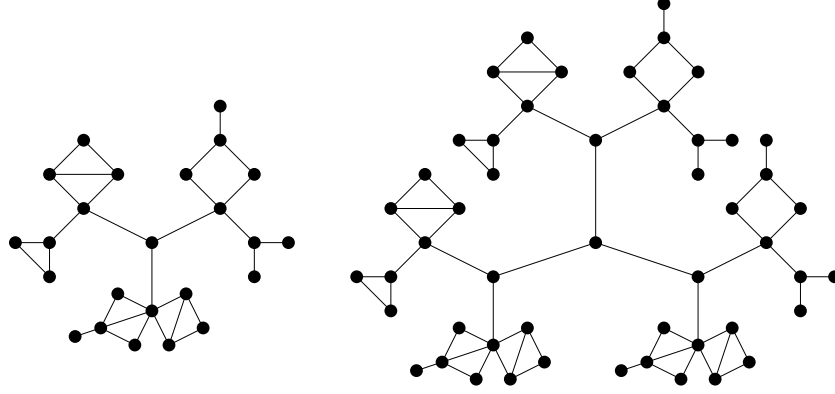


Figure 14: The left graph belongs to $\mathcal{O}_{\mathbf{Br}}(2)$ and the right to $\mathcal{O}_{\mathbf{Br}}(3)$.

Lemma 22. *Let $B \in \mathbf{Br}(k)$ and let v be its root. There does not exist a complete monotone and connected search strategy for B that uses k searchers, such that the first edge being cleaned is the trunk of B .*

Proof. We are going to prove this by induction. We can easily check that for $k = 1$ the claim holds. Let $B \in \mathbf{Br}(k)$ and let v be its root and u the other endpoint of the trunk. Since we are forced to clean B , in a connected and monotone manner, with a search strategy, say \mathcal{S} , that first cleans $\{u, v\}$, a searcher must be placed in u during each step of \mathcal{S} , therefore we must clean a $(k - 1)$ -level branch using $k - 1$ searchers that first clean the trunk of this branch, which contradicts the induction hypothesis. \square

Corollary 2. *Let $G \in \mathcal{O}_{\mathbf{Br}}(k)$, then $\text{cmms}(G) > k + 1$.*

Proof. Let $G \in \mathcal{O}_{\mathbf{Br}}(k)$. Notice that G consists of three k -level obstruction branches, say B_1 , B_2 and B_3 . If there exist a complete monotone and connected search strategy \mathcal{S} that uses $k + 1$ searchers, then from Lemma 22 \mathcal{S} cannot start by placing searchers in the central vertex, i.e. the vertex where B_1 , B_2 and B_3 are connected. Therefore, \mathcal{S} starts by placing searchers in a vertex of B_1 , B_2 or B_3 and consequently the first edge cleaned belongs to this branch. Notice that the first time that a searcher is placed on the central vertex the connectivity and monotonicity of \mathcal{S} force us to clean a k -level branch with k searchers, which is impossible according to Lemma 22. \square

Lemma 23. *Let $B \in \mathbf{Br}(k)$ and let v be its root.*

- a) *There exist a complete monotone and connected search strategy for B that uses $k + 2$ searchers, such that in each step a searcher occupies v .*
- b) *There exists a complete monotone and connected search strategy for B that uses $k + 1$ searchers, such that the first edge being cleaned is the trunk of B .*

- c) *There exist a complete monotone and connected search strategy for B that uses $k + 1$ searchers, such that the last edge being cleaned is the trunk of B .*

Proof. a) We are going to prove this by induction. We can easily check that for $k = 1$ the claim holds. Let $B \in \mathbf{Br}(k)$ and let v be its root and u the other endpoint of the trunk. We are going to describe a search strategy \mathcal{S} with the properties needed. We place a searcher in v and as second searcher in u . According to the induction hypothesis for each one of the two $(k - 1)$ -level branches connected to u there exists a complete monotone and connected search strategy that uses $k + 1$ searchers such that in each step a searcher occupies u , therefore we can continue by cleaning one of these $(k - 1)$ -level branches and then clean the other.

b) We are going to prove this by induction. For $k = 1$ the claim is trivial. Let $B \in \mathbf{Br}(k)$ and let v be its root and u the other endpoint of the trunk. There are two $(k - 1)$ -level branches connected to u , say B_1 and B_2 . The search strategy, say \mathcal{S} , with the properties needed is the following: we place a searcher in v and then slide him to u . According to the first claim of Lemma 23 there exists a complete monotone and connected search strategy \mathcal{S}_1 for B_1 that uses $k + 1$ searchers such that in each step a searcher occupies u . By the induction hypothesis there exists a complete monotone and connected search strategy \mathcal{S}_2 for B_2 that uses k searchers such that the first edge cleaned is the trunk of B_2 . Using these two search strategies we can start by cleaning B_1 , keeping in all times a searcher in u , and then we can clean B_2 .

c) We are going to prove this by induction. Notice that for $k = 1$ the claim holds. Let $B \in \mathbf{Br}(k)$ and let v be its root and u the other endpoint of the trunk. There are two $(k - 1)$ -level branches connected to u , say B_1 and B_2 . According to the induction hypothesis there exist a complete monotone and connected search strategy \mathcal{S}_1 for B_1 that uses k searchers such that the last edge cleaned is the trunk of B_1 . Moreover, according to the first claim of Lemma 23 there exists a complete monotone and connected search strategy \mathcal{S}_2 for B_2 that uses $k + 1$ searchers such that in each step a searcher occupies u . Using these two search strategies we can clean B , in a monotone and connected manner, as follows: we start by cleaning B_1 then we clean B_2 , keeping in all times a searcher in u , and then we clean $\{u, v\}$. \square

Lemma 24. *Let $G \in \mathcal{O}_{\mathbf{Br}}(k)$ and $B \in \mathbf{Br}(k)$ one of the three branches of G . If we contract an edge of B there exist a complete monotone and connected search strategy for B that uses $k + 1$ searchers, such that in each step a searcher occupies v .*

Proof. We are going to prove this by induction. It is easy to check that for $k = 1$ the claim is true. Let v be the root of B and u the other endpoint of the trunk, B_1 and B_2 the two $(k - 1)$ -level branches connected to u and $e \in E(B)$ the edge contracted. We distinguish to cases:

Case 1: $e \in E(B_1) \cup E(B_2)$. We can assume that e is an edge of B_1 . We are

going to describe a search strategy \mathcal{S} for B with the properties needed. We place a searcher in v and a second searcher in u . From the induction hypothesis there exists a complete monotone and connected search strategy \mathcal{S}_1 for B_1 that uses k searchers, such that in each step a searcher occupies u . Moreover, according to the second claim of Lemma 23 there exists a complete monotone and connected search strategy \mathcal{S}_2 for B_2 that uses k searchers such that the first edge cleaned is the trunk of B_2 . Using these two search strategies we can start by cleaning B_1 , keeping in all times a searcher in u , and then we can clean B_2 . Notice that this search strategy uses $k + 1$ searchers and during each step a searcher occupies v .

Case 2: $e = \{u, v\}$. According to the first property of Lemma 23, for each one of B_1 and B_2 there exists a complete monotone and connected search strategy that uses $k + 1$ searchers such that in each step a searcher occupies v . Hence we can clean B starting by cleaning B_1 , keeping in all times a searcher in v , and then clean B_2 . \square

Corollary 3. *If $G \in \mathcal{O}_{\text{Br}}(k)$ and G' be a contraction of G , then $\text{cmms}(G') = k + 1$.*

Proof. It suffices to prove this claim for a single edge contraction. Let $G \in \mathcal{O}_{\text{Br}}(k)$, let B_1 , B_2 , and B_3 be the three k -level obstruction-branches of G connected to v , $e \in E(G)$ the edge contracted and G' the graph obtained from G after the contraction of e . We can assume that $e \in E(B_2)$. We are going to describe a complete monotone and connected search strategy \mathcal{S} for G . From the third claim of Lemma 23 we know that there exist a complete monotone and connected search strategy \mathcal{S}_1 for B_1 that uses $k + 1$ searchers, such that the last edge cleaned is the trunk of B_1 . From Lemma 24 we know that there exist a complete monotone and connected search strategy \mathcal{S}_2 for B_2 that uses $k + 1$ searchers, such that in each step a searcher occupies the root of B_2 , in other words v . From the second claim of Lemma 23 we know that there exist a complete monotone and connected search strategy \mathcal{S}_3 for B_3 that uses $k + 1$ searchers, such that the first edge cleaned is the trunk of B_3 . Therefore, we can clean G' starting by cleaning B_1 according to \mathcal{S}_1 (notice that the trunk of B_1 will be the last edge of $E(B_1)$ being cleaned), then clean B_2 according to \mathcal{S}_2 , keeping in all times a searcher in v , and finish by cleaning B_3 according to \mathcal{S}_3 . \square

Combining Corollaries 2 and 3 we conclude that every graph in $\mathcal{O}_{\text{Br}}(k)$ is a contraction obstruction for the graph class $\mathcal{G}[\text{cmms}, k + 1]$ and therefore $\mathcal{O}_{\text{Br}}(k) \subseteq \text{obs}_{\leq}(\mathcal{G}[\text{cmms}, k + 1])$.

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